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# A KEY TO DAY'S ALGEBRA.

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# CONTENTS.

Page

ADDITION, 7

SUBTRACTION, 8

MULTIPLICATION, 9

Multiplication by Detached Coefficients, 11

DIVISION, 13

Division by Detached Coefficients, 17

Resolving Polynomials into Factors, 18

FRACTIONS, 19

SIMPLE EQUATIONS, 22

Simple Equations containing Two Unknown Quantities, 36

Simple Equations containing Three or More Unknown Quantities, 43

RADICAL QUANTITIES, 46

Reduction of Radical Quantities, 46

Multiplication of Radical Quantities, 47

REDUCTION OF EQUATIONS BY INVOLUTION AND EVOLUTION, 51

QUADRATIC EQUATIONS, 57

PROPORTION, 71

ARITHMETICAL PROGRESSION, 75

GEOMETRICAL PROGRESSION, 81

GREATEST COMMON MEASURE, 85

INVOLUTION OF BINOMIALS, 88

EVOLUTION OF COMPOUND QUANTITIES, 97

ROOTS OF BINOMIAL SURDS, 100

INFINITE SERIES, 101

Indeterminate Coefficients, 103

Recurring Series, 109

Method of Differences, 110

CONTINUED FRACTIONS, 114

INTERPOLATION, 117

Page
GENERAL PROPERTIES OF EQUATIONS, ........ 118
Sturm's Theorem, ........ 143
Elimination, ........ 148
RESOLUTION OF EQUATIONS, ........ 153
Approximation by Double Position, ........ 157
Newton's Method, ........ 161
Horner's Method, ........ 164
APPLICATION OF ALGEBRA TO GEOMETRY, ........ 175

# KEY TO DAY'S ALGEBRA.

## ADDITION.

**Ex. 1.**
$$\begin{array}{llllll} \text{To} & 7 & -xy & & & \\ \text{Add} & 2 & & +ab & & \\ \text{``} & -6 & +3xy & & -8h & \\ \hline \textit{Ans.} & 3 & +2xy & +ab & -8h. & \end{array}$$

**Ex. 2.**
$$\begin{array}{llllll} \text{To} & 2y^2 & -5x & -2ay & & \\ \text{Add} & -2y^2 & & & +bn-5 & (=bn-6-2y^2+1) \\ \hline \textit{Ans.} & & -5x & -2ay & +bn-5. & \end{array}$$

**Ex. 3.**
$$\begin{array}{lllll} \text{To} & bc & +2x^2 & -axy & \\ \text{Add} & & -x^2 & & +9-h \\ \text{``} & & -4x^2 & & -7+6h \\ \hline \textit{Ans.} & bc & -3x^2 & -axy & +2+5h. \end{array}$$

**Ex. 4.**
$$\begin{array}{lllll} \text{To} & 5y^3 & -8 & +ah & (=5y^3-9+ah+1) \\ \text{Add} & -y^3 & -10 & +12ah & \\ \hline \textit{Ans.} & 4y^3 & -18 & +13ah. & \end{array}$$

**Ex. 5.**
$$\begin{array}{llllll} \text{To} & bx^2 & +4 & -y & -3a^2x \\ \text{Add} & 9bx^2 & -13 & +y & +3a^2x \\ \hline \textit{Ans.} & 10bx^2 & -9. & & \end{array}$$

Ex. 6. To $4hn^2+3ab+x$ $(=5hn^2+3ab-hn^2+x)$
Add $hn^2-7ab-x$
*Ans.* $5hn^2-4ab$.

Ex. 7. To $4x^2-3bx-y$
Add $x^2+bx-ay-5n$
*Ans.* $5x^2+2bx-y-ay-5n$.

Ex. 8. To $ah+2mn-3x$
Add $-2ah+4x+4an$
*Ans.* $-ah+2mn+x+4an$.

## SUBTRACTION.

### ARTICLE 80. EXAMPLE 4.

From $3x^2-2ay^2+bc$
Sub. $-2x^2+ay^2-4bc$ $(=x^2-4bc+ay^2-3x^2)$
*Ans.* $5x^2-3ay^2+5bc$.

### ARTICLE 82. EXAMPLE 4.

From $5x-y^2+hn$
Sub. $5x-4y^2+2hn+ah-ny$ $(=2hn-4y^2-x+ah+6x-ny)$
*Ans.* $3y^2-hn-ah+ny$.

### EXAMPLE 5.

From $2x^2+ay-12$
Sub. $-x^2-3ay-12+2h-c$ $(=1-x^2+2h-13-3ay-c)$
*Ans.* $3x^2+4ay-2h+c$.

Ex. 6. From $ab-3ch-2$
Sub. $-5ab-ch+3-d+y$
*Ans.* $6ab-2ch-5+d-y$.

## MULTIPLICATION.

ART. 97. Ex. 5.

$$\begin{array}{ll} \text{Mult.} & ax+b^2+c \\ \text{By} & \underline{n+3+7a^2} \\ \textit{Prod.} & anx+b^2n+cn+3ax+3b^2+3c+7a^2x+7a^2b^2+7a^2c. \end{array}$$

Ex. 6.

$$\begin{array}{ll} \text{Mult.} & 1+x+2hy \\ \text{By} & \underline{5a+3b+6} \\ \textit{Prod.} & 5a+5ax+10ahy+3b+3bx+6bhy+6+6x+12hy. \end{array}$$

ART. 98. Ex. 4.

$$\begin{array}{llllll} \text{Mult.} & h+a+1 & & & & \\ \text{By} & \underline{3h+2a+5} & & & & \\ & 3h^2+3ah+3h & & & & \\ & \quad +2ah & & +2a^2+2a & & \\ & & +5h & & +5a+5 & \\ \textit{Prod.} & 3h^2+5ah+8h+2a^2+7a+5. & & & & \end{array}$$

Ex. 5.

$$\begin{array}{ll} \text{Mult.} & 2a+3+xy \\ \text{By} & \underline{a+2+3xy} \\ & 2a^2+3a+axy \\ & \quad +4a \qquad +6+2xy \\ & \qquad \underline{+6axy \quad +9xy+3x^2y^2} \\ \textit{Prod.} & 2a^2+7a+7axy+6+11xy+3x^2y^2. \end{array}$$

Ex. 6.

$$\begin{array}{lll} \text{Mult.} & c+3d+2e & \\ \text{By} & \underline{2bdx} & (=b\times 2d\times x) \\ \textit{Prod.} & 2bcdx+6bd^2x+4bdex. & \end{array}$$

ART. 102. Ex. 4.

$$\begin{array}{ll} \text{Mult.} & b^2-5-2cx \\ \text{By} & \underline{7-b-cy} \\ \textit{Prod.} & 7b^2-35-14cx-b^3+5b+2bcx-b^2cy+5cy+2c^2xy. \end{array}$$

$$\begin{array}{lll} \text{Ex. 5.} & \text{Mult.} & 3ax+h-2 \\ & \text{By} & \underline{5y^2-n+1} \end{array}$$

*Prod.* $15axy^2+5hy^2-10y^2-3anx-hn+2n+3ax+h-2.$

**Art. 107.** Ex. 1.

$$\begin{array}{llrrrrr} \text{Mult.} & 3a & - & x & +2 & & \\ \text{By} & a & - & 4x & -1 & & \\ \hline & 3a^2 & - & ax & +2a & & \\ & & -12ax & & & +4x^2 & -8x \\ & & & & -3a & & +x-2 \\ \hline \textit{Prod.} & 3a^2 & -13ax & - & a & +4x^2 & -7x-2. \end{array}$$

$$\begin{array}{lll} \text{Ex. 2.} & \text{Mult.} & 15axy=(xy\times 3a\times 5) \\ & \text{By} & \underline{2n+1-3h^2} \\ & \textit{Prod.} & 30anxy+15axy-45ah^2xy. \end{array}$$

$$\begin{array}{llll} \text{Ex. 3.} & \text{Mult.} & 5x-15ab & =5(x-3ab) \\ & \text{By} & 30hy & =(y\times 2\times 5h\times 3) \\ \hline & \textit{Prod.} & 150hxy-450abhy. & \end{array}$$

$$\begin{array}{lll} \text{Ex. 4.} & \text{Mult.} & 6a^2+3bc-3=3(2a^2+bc-1) \\ & \text{By} & \underline{5d-2dx \quad =d(6 \quad -2x-1)} \end{array}$$

*Prod.* $30a^2d+15bcd-15d-12a^2dx-6bcdx+6dx.$

$$\begin{array}{lll} \text{Ex. 5.} & \text{Mult.} & ax+5x-h-2 \\ & \text{By} & \underline{cx+dx-ch-dh=(c+d)\,(x-h)} \end{array}$$

*Prod.* $acx^2+5cx^2-chx-2cx+adx^2+5dx^2-dhx-2dx-achx-5chx+ch^2+2ch-adhx-5dhx+dh^2+2dh.$

Or by Reduction. $acx^2+5cx^2-6chx-2cx+adx^2+5dx^2-6dhx-2dx-achx+ch^2+2ch-adhx+dh^2+2dh.$

$$\begin{array}{lll} \text{Ex. 6.} & \text{Mult.} & 5x^2y-b+3c \quad =5x^2y-(b-3c) \\ & \text{By} & \underline{bn \quad -n+ \quad b-1=(b-1)\,(n+1).} \end{array}$$

*Prod.* $5bnx^2y-b^2n+3bcn-5nx^2y+bn-3cn+5bx^2y-b^2+3bc-5x^2y+b-3c.$

$$\begin{array}{lll}
\text{Ex. 7.} & \text{Mult.} & ay-7+bc-bm \\
 & \text{By} & \underline{-2n+1-3x \qquad =-(2n-1+3x)} \\
 & \textit{Prod.} & -2any+14n-2bcn+2bmn+ay-7+bc \\
 & & \quad -bm-3axy+21x-3bcx+3bmx.
\end{array}$$

---

## Multiplication by Detached Coefficients.

Ex. 1. Mult. $b^3-b^2x+bx^2-x^3$

By $b+x$

$$\begin{array}{lr}
\text{Coefficients of Multiplicand,} & 1-1+1-1\phantom{-1} \\
\text{" \qquad Multiplier,} & \underline{1+1\phantom{-1+1-1}} \\
 & 1-1+1-1\phantom{-1} \\
 & \underline{\phantom{1}\;1-1+1-1} \\
\text{Coefficients of Product,} & 1 \quad * \quad * \quad *-1
\end{array}$$

Each term of the product has four literal factors. The first term cannot contain $x$, nor the last term $b$; hence the product is $b^4-x^4$.

Ex. 2. Mult. $y^3-3y^2+3y-1$

By $y^2-1$

$$\begin{array}{lr}
\text{Coefficients of Multiplicand,} & 1-3+3-1\phantom{-3+1} \\
\text{" \qquad Multiplier,} & \underline{1\pm0-1\phantom{+3-1-3+1}} \\
 & 1-3+3-1\phantom{-3+1} \\
 & \underline{\phantom{1-3}-1+3-3+1} \\
\text{Coefficients of Product,} & 1-3+2+2-3+1
\end{array}$$

The exponent of $y$ in the first term of the product will be 5, and will decrease by 1 in the successive terms; hence the product is $y^5-3y^4+2y^3+2y^2-3y+1$.

Ex. 3. Mult. $x^4+2x^2y^2+y^4$

By $x^2-2xy+y^2$

Coefficients of Multiplicand, $1+0+2+0+1$
" Multiplier, $1-2+1$

$$\begin{array}{r}1+0+2+0+1\phantom{-0-2+0+1}\\ -2-0-4-0-2\phantom{+0+1}\\ 1+0+2+0+1\\ \hline\end{array}$$

Coefficients of Product, $1-2+3-4+3-2+1$

Each term of the product must have six literal factors: the first term cannot contain $y$, and the intervening terms will contain it 1, 2, 3, &c., times, according to their distance from the first; the last term cannot contain $x$, and the intervening terms will contain it 1, 2, 3, &c., times, according to their distance from the last. Hence the product is

$$x^6-2x^5y+3x^4y^2-4x^3y^3+3x^2y^4-2xy^5+y^6.$$

Ex. 4. Mult. $a^2-1$
By $a^2+1$
Coefficients of Multiplicand, $1\pm0-1$
" Multiplier, $1\pm0+1$

$$\begin{array}{r}1\pm0-1\phantom{\pm0-1}\\ 1\pm0-1\\ \hline 1\ \ *\ \ *\ \ *-1\end{array}$$

The first term of the product will contain $a$ four times; and the last will not contain it at all. Hence the product is $a^4-1$.

## ARTICLE 109. THEOREM I.

Ex. 3. $(3x^3+1)^2=9x^6+6x^3+1$.
Ex. 4. $(100+1)^2=10000+200+1$.
Ex. 5. $(90+5)^2=8100+900+25$.
Ex. 6. $(5a^2y+2ax^2)^2=25a^4y^2+20a^3x^2y+4a^2x^4$.
Ex. 7. $(4an^3+7)^2=16a^2n^6+56an^3+49$.
Ex. 8. $(12+\frac{1}{2})^2=144+12+\frac{1}{4}$.

### THEOREM II.

Ex. 3. $(1000-3)^2=1000000-6000+9$.

Ex. 4. $(5x^2-2y)^2=25x^4-20x^2y+4y^2$.

Ex. 5. $(1-2x^3)^2=1-4x^3+4x^6$.

Ex. 6. $(3a^2b-4ax^3)^2=9a^4b^2-24a^3bx^3+16a^2x^6$.

Ex. 7. $(20-\frac{1}{10})^2=400-4+\frac{1}{100}$.

Ex. 8. $(2x^2-\frac{1}{2})^2=4x^4-2x^2+\frac{1}{4}$.

Ex. 9. $(\frac{1}{10}-\frac{1}{100})^2=\frac{1}{100}-\frac{1}{500}+\frac{1}{10000}$.

### THEOREM III.

Ex. 3. $(a^2x+7x^2y)\,(a^2x-7x^2y)=a^4x^2-49x^4y^2$.

Ex. 4. $(2x^3y+1)\,(2x^3y-1)=4x^6y^2-1$.

Ex. 5. $(1+4a^2)\,(1-4a^2)=1-16a^4$.

Ex. 6. $(x^2+7xy)\,(x^2-7xy)=x^4-49x^2y^2$.

Ex. 7. $(10+\frac{1}{10})\,(10-\frac{1}{10})=100-\frac{1}{100}$.

Ex. 8. $(100+7)\,(100-7)=10000-49$.

◆

## DIVISION.

### ARTICLE 127. EXAMPLE 4.

$$
\begin{array}{l|l}
x^4-x^3+ax^3-ax^2+7xy^2-7y^2 & x-1. \\
\underline{x^4-x^3} \qquad\qquad\qquad\qquad \textit{Quot.} & \overline{x^3+ax^2+7x^2} \\
\qquad\quad ax^3-ax^2 & \\
\qquad\quad \underline{ax^3-ax^2} & \\
\qquad\qquad\qquad 7xy^2-7y^2 & \\
\qquad\qquad\qquad \underline{7xy^2-7y^2} &
\end{array}
$$

Ex. 5.

$$
\begin{array}{l|l}
6a^2h-4a^2n+2a^2x+3h-2n+x & 3h-2n+x \\
\underline{6a^2h-4a^2n+2a^2x\qquad\qquad\qquad} & \overline{2a^2+1,\ \textit{Quot.}} \\
\qquad\qquad\qquad\qquad 3h-2n+x & \\
\qquad\qquad\qquad\qquad \underline{3h-2n+x} &
\end{array}
$$

**Ex. 6.**
$$\begin{array}{r|l}
3x^4+5x^3y-9x^2y^2+7xy^3-6y^4 & x^2+2xy-3y^2 \\ \cline{2-2}
3x^4+6x^3y-9x^2y^2\phantom{+7xy^3-6y^4} & 3x^2-xy+2y^2,\ \textit{Quot.} \\ \cline{1-1}
-x^3y\phantom{-2x^2y^2}+7xy^3\phantom{-6y^4} & \\
-x^3y-2x^2y^2+3xy^3\phantom{-6y^4} & \\ \cline{1-1}
2x^2y^2+4xy^3-6y^4 & \\
2x^2y^2+4xy^3-6y^4 & \\ \cline{1-1}
\end{array}$$

### Example 11.

$$\begin{array}{r|l}
3ab-6ax-bh+2hx+2b^2-4bx+c & b-2x \\ \cline{2-2}
3ab-6ax\phantom{-bh+2hx+2b^2-4bx+c} & \textit{Quot.}\ 3a-h+2b+\frac{c}{b-2x}. \\ \cline{1-1}
-bh+2hx\phantom{+2b^2-4bx+c} & \\
-bh+2hx\phantom{+2b^2-4bx+c} & \\ \cline{1-1}
2b^2-4bx\phantom{+c} & \\
2b^2-4bx\phantom{+c} & \\ \cline{1-1}
c & \\
\end{array}$$

**Ex. 12.**
$$\begin{array}{r|l}
6a^3+4a^2-3ax-2x+3a+7 & 3a+2 \\ \cline{2-2}
6a^3+4a^2\phantom{-3ax-2x+3a+7} & \textit{Quot.}\ 2a^2-x+1+\frac{5}{3a+2}. \\ \cline{1-1}
-3ax-2x\phantom{+3a+7} & \\
-3ax-2x\phantom{+3a+7} & \\ \cline{1-1}
3a+7 & \\
3a+2 & \\ \cline{1-1}
5 & \\
\end{array}$$

---

### Promiscuous Examples.

**Ex. 1.**
$$\begin{array}{ll}
\text{Div.} & 5x^2y-10ax^3+15bxy^2-5x \\
\text{By} & 5x \\ \cline{2-2}
\textit{Quot.} & xy-\ 2ax^2+\ 3by^2-1. \\
\end{array}$$

**Ex. 2.**
$$\begin{array}{ll}
\text{Div.} & 12h-15a+3y^2-3-9bcy+2x \\
\text{By} & 3 \\ \cline{2-2}
\textit{Quot.} & 4h-\ 5a+\ y^2-1-3bcy+\frac{2x}{3}. \\
\end{array}$$

$$\begin{array}{lll} \text{Ex. 3.} & \text{Div.} & (2+ax)\,(3-c)\,y \\ & \text{By} & \underline{(2+ax)\,(3-c)} \\ & \textit{Quot.} & y. \end{array}$$

$$\text{Ex. 4.}\quad \begin{array}{l|l} x^2y-3xy-x+2xy^2 & \underline{xy-3y-1+2y^2} \\ \underline{x^2y-3xy-x+2xy^2} & \text{Quot. } x. \end{array}$$

$$\begin{array}{lll} \text{Ex. 5.} & \text{Div.} & -b+b^2h-by+2a^2h+nx-3 \\ & \text{By} & \underline{-b} \\ & \textit{Quot.} & 1-bh+y+\dfrac{2a^2h+nx-3}{-b}. \end{array}$$

$$\begin{array}{lll} \text{Ex. 6.} & \text{Div.} & bnx-2n^2-ahn-4n+x \\ & \text{By} & \underline{-n^2x} \\ & \textit{Quot.} & -\dfrac{b}{n}+\dfrac{2}{x}+\dfrac{ah}{nx}+\dfrac{4}{nx}-\dfrac{1}{n^2}. \end{array}$$

$$\begin{array}{lll} \text{Ex. 7.} & \text{Div.} & 2b^2-bx^2y-3bx+x-9 \\ & \text{By} & \underline{3bx^2y} \\ & \textit{Quot.} & \dfrac{2b}{3x^2y}-\dfrac{1}{3}-\dfrac{1}{xy}+\dfrac{1}{3bxy}-\dfrac{3}{bx^2y}. \end{array}$$

$$\begin{array}{lll} \text{Ex. 8.} & \text{Div.} & 4a^3x^2-a^2-2a^2x+4(=4a^3x^2-a^2-2a^2x+6-2) \\ & \text{By} & \underline{-2a^3} \\ & \textit{Quot.} & -2x^2+\dfrac{1}{2a}+\dfrac{x}{a}+\dfrac{2}{a^3}. \end{array}$$

$$\text{Ex. 9.}\quad \begin{array}{r|l} b+x+bnx+nx^2 & \underline{b+x} \\ \underline{b+x\qquad\qquad\quad} & \quad 1+nx. \\ +bnx+nx^2 & \\ \underline{+bnx+nx^2} & \\ 2\qquad & \end{array}$$

Ex. 10.

$$\begin{array}{l|l}
1-3x+3x^2-x^3 & 1-x \\ \cline{2-2}
\underline{1-x\phantom{+3x^2-x^3}} & 1-2x+x^2 \\
\quad -2x+3x^2 & \\
\quad \underline{-2x+2x^2\phantom{-x^3}} & \\
\qquad\qquad x^2-x^3 & \\
\qquad\qquad \underline{x^2-x^3} &
\end{array}$$

Ex. 11.

$$\begin{array}{l|l}
b^3+3b^2x+3bx^2+x^3 & b+x \\ \cline{2-2}
\underline{b^3+\ b^2x\phantom{+3bx^2+x^3}} & b^2+2bx+x^2. \\
\quad 2b^2x+3bx^2 & \\
\quad \underline{2b^2x+2bx^2\phantom{+x^3}} & \\
\qquad\qquad bx^2+x^3 & \\
\qquad\qquad \underline{bx^2+x^3} &
\end{array}$$

Ex. 12.

$$\begin{array}{l|l}
3x^3+16x^2-5x+11 & x+5 \\ \cline{2-2}
\underline{3x^3+15x^2\phantom{-5x+11}} & 3x^2+x-10+\dfrac{61}{x+5}. \\
\qquad x^2-5x & \\
\qquad \underline{x^2+5x\phantom{+11}} & \\
\qquad\quad -10x+11 & \\
\qquad\quad \underline{-10x-50} & \\
\qquad\qquad\qquad 61 &
\end{array}$$

Ex. 13.

$$\begin{array}{l|l}
1-a^5 & 1-a \\ \cline{2-2}
\underline{1-a\phantom{^5}} & 1+a+a^2+a^3+a^4. \\
\quad a-a^5 & \\
\quad \underline{a-a^2} & \\
\qquad a^2-a^5 & \\
\qquad \underline{a^2-a^3} & \\
\qquad\quad a^3-a^5 & \\
\qquad\quad \underline{a^3-a^4} & \\
\qquad\qquad a^4-a^5 & \\
\qquad\qquad \underline{a^4-a^5} &
\end{array}$$

Ex. 14.
$$\begin{array}{l|l}
x^4-6x^2y^2+5y^4 & x-3y \\ \cline{2-2}
\underline{x^4-3x^3y} & x^3+3x^2y+3xy^2+9y^3+\dfrac{32y^4}{x-3y}. \\
\quad 3x^3y-6x^2y^2 & \\
\quad \underline{3x^3y-9x^2y^2} & \\
\qquad 3x^2y^2+5y^4 & \\
\qquad \underline{3x^2y^2-9xy^3} & \\
\qquad\quad 9xy^3+\ 5y^4 & \\
\qquad\quad \underline{9xy^3-27y^4} & \\
\qquad\qquad 32y^4 &
\end{array}$$

Ex. 15.
$$\begin{array}{l|l}
6a^4-10a^3+4a-15 & 3a^2-2a+1 \\ \cline{2-2}
\underline{6a^4-\ 4a^3+2a^2} & 2a^2-2a-2+\dfrac{2a-13}{3a^2-2a+1}. \\
\quad -\ 6a^3-2a^2+4a & \\
\quad \underline{-\ 6a^3+4a^2-2a} & \\
\qquad -6a^2+6a-15 & \\
\qquad \underline{-6a^2+4a-\ 2} & \\
\qquad\qquad 2a-13 &
\end{array}$$

---

## Division by Detached Coefficients.

Ex. 1. Div. $a^4-a^2x^2+2ax^3-x^4$, by $a^2-ax+x^2$.

$$\begin{array}{ll|ll}
\text{Coeff. of Div'd,} & 1+0-1+2-1 & 1-1+1 & \text{Coeff. of Div'r.} \\ \cline{3-3}
& \underline{1-1+1} & 1+1-1 & \text{“} \qquad \text{Quot.} \\
& \quad 1-2+2 & & \\
& \quad \underline{1-1+1} & & \\
& \qquad -1+1-1 & & \\
& \qquad \underline{-1+1-1} & &
\end{array}$$

The first term of the quotient must be $a^2$; and hence the letters with their exponents in the following terms $ax$ and $x^2$. Therefore the quotient is $a^2+ax-x^2$.

Ex. 2. Div. $x^4-4x^3+6x^2-4x+1$, by $x-1$.

Coeff. of Div'd, $1-4+6-4+1\ |\ \underline{1-1}$ Coeff. of Div'r.

$\underline{1-1}$ $\quad 1-3+3-1$ " Quot.

$-3+6$

$\underline{-3+3}$ $\quad x^3-3x+3x-1$, *Quot.*

$3-4$

$\underline{3-3}$

$-1+1$

$\underline{-1+1}$

Ex. 3. Div. $x^5+x^4y-5x^3y^2+6xy^4+2y^5$, by $x^2+3xy+y^2$.

Coeff. of Div'd, $1+1-5-0+6+2\ |\ \underline{1+3+1}$ Coeff. of Div'r.

$\underline{1+3+1}$ $\quad 1-2+2$

$-2-6-0$

$\underline{-2-6-2}$ $\quad x^3-2x^2y+2y^3$, *Quot.*

$2+6+2$

$2+6+2$

Ex. 4. Div. $1-4b+10b^2-16b^3+17b^4-12b^5$, by $1-2b+3b^2$.

Coeff. of Div'd, $1-4+10-16+17-12\ |\ \underline{1-2+3}$ Coeff. of

$\underline{1-2+3}$ $\quad 1-2+3-4$ [Div'r.

$-2+7-16$

$\underline{-2+4-6}$ $\quad 1-2b+3b^2-4b^3$, *Quot.*

$3-10+17$

$\underline{3-6+9}$

$-4+8-12$

$-4+8-12$

## Resolving Polynomials into Factors.

Ex. 5. $4x^3+12x^2y+9xy^2=x(4x^2+12xy+3y^2)=x(2x+3y)^2=$ $x(2x+3y)\ (2x+3y)$.

Ex. 6. $x^3+y^3=(x+y)\ (x^2-xy+y^2)$.

Ex. 7. $1-a^3=(1-a)\ (1+a+a^2)$.

Ex. 8. $a^4-b^4=(a^2+b^2)\ (a^2-b^2)=(a^2+b^2)\ (a+b)\ (a-b)$.

Ex. 9. $y^6+1=(y^2+1)\ (y^4-y^2+1)$.

Ex. 10. $4x^3-4x^2+x=x(2x-1)^2=x(2x-1)\ (2x-1)$.

Ex. 11. $abh^2-2abh+ab=ab(h-1)^2=ab(h-1)\ (h-1)$.

Ex. 12. $63nx^2y^2+84nxy+28n=7n(9x^2y^2+12xy+4)=7n(3xy+2)^2=7n(3xy+2)\ (3xy+2)$.

Ex. 13. $9h-h=h(3-1)\ (3+1)$.

Ex. 14. $3^8-1=(3^4+1)\ (3^4-1)=(3^4+1)\ (3^2+1)\ (3^2-1)=(3^4+1)\ (3^2+1)\ (3+1)\ (3-1)$.

Ex. 15. $25h^2n^4-\frac{1}{4}=(5hn^2-\frac{1}{2})\ (5hn^2+\frac{1}{2})$.

◆

## FRACTIONS.

Art. 50. Ex. 2. Reduce $\frac{ab}{2h}$, and $\frac{3n^2}{m}$, and $\frac{g}{3x}$.

$$\left.\begin{array}{l} ab\times 3x\times m=\ \ 3abmx \\ 3n^2\times 2h\times 3x=18hn^2x \\ 2h\times m\times g=\ \ 2ghm \end{array}\right\}\text{The numerators.}$$

$$2h\times m\times 3x=\ 6hmx\quad\text{The common denominator.}$$

The fractions reduced are $\frac{3abmx}{6hmx}$, and $\frac{18hn^2x}{6hmx}$, and $\frac{2ghm}{6hmx}$.

Ex. 3. Reduce $\frac{1}{4}$, and $\frac{2}{y^2}$, and $\frac{1-a}{x+2}$.

$$\left.\begin{array}{l} 1\times y^2\times(x+2)=\ xy^2+\ 2y^2 \\ 4\times 2\ \times(x+2)=8x\ \ +16 \\ 4\times y^2\times(1-a)=4y^2\ -\ 4ay^2 \end{array}\right\}\text{The numerators.}$$

$$4\times y^2\times(x+2)=4xy^2+\ 8y^2\quad\text{The common denominator.}$$

The fractions reduced are $\frac{xy^2+2y^2}{4xy^2+8y^2}$, $\frac{8x+16}{4xy^2+8y^2}$, $\frac{4y^2-4ay^2}{4xy^2+8y^2}$.

Ex. 4. Reduce $\frac{2}{x+2}$, and $\frac{2}{x-2}$.

$$\left.\begin{array}{r}2\times(x-2)=2x-4\\2\times(x+2)=2x+4\end{array}\right\}\text{ The numerators.}$$

$$(x+2)\ (x-2)=x^2-4 \quad \text{The common denominator.}$$

The fractions reduced are $\frac{2x-4}{x^2-4}, \frac{2x+4}{x^2-4}$.

**Art. 154.** Ex. 4. Reduce $n-\frac{h}{3}$.

$$n-\frac{h}{3}=\frac{n}{3}-\frac{h}{3}=\frac{n-h}{3}, \quad \textit{Ans.}$$

Ex. 5. Reduce $1+\frac{2x^2-1}{5x}$.

$$1+\frac{2x^2-1}{5x}=\frac{5x+2x^2-1}{5x}, \quad \textit{Ans.}$$

Ex. 6. Reduce $2y-\frac{3x}{y-2a}$.

$$2y-\frac{3x}{y-2a}=\frac{2y^2-2ay-3x}{y-2a}, \quad \textit{Ans.}$$

**Art. 155.** Ex. 6. From $\frac{1}{a}$ sub. $\frac{x}{2}$.

Red. to Com. Denom. $\frac{2}{2a}-\frac{ax}{2a}=\frac{2-ax}{2a}$, *Ans.*

Ex. 7. From $\frac{x+2}{y}$ sub. $\frac{y-2}{3x}$.

Red. to Com. De'm. $\frac{3x^2+6x}{3xy}-\frac{y^2-2y}{3xy}=\frac{3x^2-y^2+6x+2y}{3xy}$, *Ans.*

**Art. 157.** Ex. 4. From $2y+x-\frac{1-2a}{3}$ sub. $y-2x+\frac{a-2}{2}$.

$$2y+x-\frac{1-2a}{3}=\frac{6y+3x-1+2a}{3},$$

$$y-2x+\frac{a-2}{2}=\frac{2y-4x+a-2}{2}$$

Reduced to Common Denominators.

$$\frac{12y+6x-2+4a}{6}-\frac{6y-12x+3a-6}{6}=\frac{6y+18x+a+4}{6}, \text{ Ans.}$$

ART. 158. Ex. 4. Mult. $\frac{2-x}{1+y}$ by $\frac{a-b}{y-2}$.

$$\frac{2-x}{1+y}\times\frac{a-b}{y-2}=\frac{2a-2b-ax+bx}{-y+y^2-2}, \text{ Ans.}$$

Ex. 5. Mult. $\frac{2}{3-n}$ by $\frac{1}{5}$.

$$\frac{2}{3-n}\times\frac{1}{5}=\frac{2}{15-5n}, \text{ Ans.}$$

ART. 159. Ex. 3. Mult. $\frac{1}{3-n}$, $\frac{h}{2}$, and $\frac{n+2}{x}$.

$$\frac{1}{3-n}\times\frac{h}{2}\times\frac{n+2}{x}=\frac{hn+2h}{6x-2nx}, \text{ Prod.}$$

Ex 4. Mult. $\frac{3x}{y^2}$, $\frac{5}{2}$, and $\frac{1+2b}{4-a}$.

$$\frac{3x}{y^2}\times\frac{5}{2}\times\frac{1+2b}{4-a}=\frac{15x+30bx}{8y^2-2ay^2}, \text{ Prod.}$$

ART. 165. Ex. 5. Div. $\frac{2a^2-x}{2h}$ by $\frac{a^2-2x}{n}$.

$$\frac{2a^2-x}{2h}\times\frac{n}{a^2-2x}=\frac{2a^2n-nx}{2a^2h-4hx}, \text{ Quot.}$$

Ex. 6. Div. $\frac{ab+1}{3}$ by $\frac{2}{ab-1}$.

$$\frac{ab+1}{3}\times\frac{ab-1}{2}=\frac{a^2b^2-1}{6}, \text{ Quot.}$$

## SIMPLE EQUATIONS.

ART. 192. Ex. 3. Reduce $24+5x-30=159-6x$.

Transposing, $5x+6x=159+30-24$

Uniting terms, $11x=165$

Dividing by 11, $x=\frac{165}{11}=15$, *Ans.*

Ex. 4. Reduce $\frac{2x-7}{3}-5=\frac{x}{4}-\frac{x-13}{2}$.

Clearing of fractions, $8x-28-60=3x-6x+78$

Transposing, $8x+6x-3x=60+28+78$

Uniting terms, $11x=166$

Dividing by 11, $x=\frac{166}{11}$.

Ex. 5. Reduce $3x+8+\frac{3}{2}x=7x-22$.

Clearing of fractions, $6x+16+3x=14x-44$

Trans. and unit. terms, $5x=60$, and $x=12$.

Ex. 6. Reduce $5\frac{3}{4}-\frac{x}{2}=2x-16\frac{3}{4}$.

Clearing of fractions, $23-2x=8x-67$

Trans. and unit. terms, $10x=90$, or $x=9$.

Ex. 7. Reduce $\frac{3x}{5}-\frac{7x}{10}+\frac{3x}{4}=\frac{7x}{8}-15$.

The least common multiple of denom. $=40$

Clearing of fractions, $24x-28x+30x=35x-600$

Trans. and unit. terms. $9x=600$, and $x=\frac{600}{9}=66\frac{2}{3}$.

Ex. 8. Reduce $\frac{x}{2}+\frac{x}{3}-\frac{x}{5}=7x-\frac{x}{4}-734$.

Mult. by 60, . . . . $30x+20x-12x=420x-15x-44040$

Trans. and unit. terms, $367x=44040$; and $x=120$.

Ex. 9. Reduce $\frac{31}{6}x+134-\frac{13}{3}x=11x+\frac{3}{8}x+7\frac{1}{2}$.

Mult. by 24, $124x+3216-104x=264x+9x+180$
Trans. and unit. terms, $253x=3036$, and $x=12$.

Ex. 10. Reduce $\frac{ax-10}{3}=\frac{x}{2}-\frac{b-5}{3}$.

Mult. by 6, $2ax-20=3x-2b+10$
Transposing, $2ax-3x=30-2b$
Div. by $(2a-3)$, $x=\frac{30-2b}{2a-3}$.

Ex. 11. Reduce $\frac{x+3}{a}=\frac{x}{b}+12$.

Mult. by $ab$, $bx+3b=ax+12ab$
Trans. and div. by $(b-a)$, $bx-ax=12ab-3b$, & $x=\frac{3b(4a-1)}{b-a}$.

Ex. 12. Reduce $\frac{a-1}{x}-1=b$.

Mult. by $x$, $a-1-x=bx$
Trans. and div. by $(1+b)$, $x+bx=a-1$, and $x=\frac{a-1}{1+b}$.

Ex. 13. Reduce $13-\frac{3}{2x-1}=11$.

Mult. by $(2x-1)$, $26x-13-3=22x-11$
Trans. and unit. terms, $4x=5$, and $x=\frac{5}{4}$.

Ex. 14. Reduce $\frac{x}{2}-\frac{x}{3}=x-21$.

Mult. by 6, $3x-2x=6x-126$
Trans. and unit. terms, $5x=126$ Div. $x=\frac{126}{5}=25\frac{1}{5}$.

Ex. 15. Reduce $\frac{x}{3}-\frac{x}{4}=\frac{x}{2}-\frac{5}{12}$.

Mult. by 12, $4x-3x=6x-5$
Trans. and unit. terms, $5x=5$ Div. $x=1$.

Ex. 16. Reduce $\frac{2x+5}{3}-12=\frac{x}{2}-\frac{x-7}{4}$.

Mult. by 12, $8x+20-144=6x-3x+21$
Trans. and unit. terms, $5x=145$ Div. $x=29$.

Ex. 17. Reduce $\frac{x+7}{2}-\frac{2x-7}{3}=\frac{x}{4}-15$.

Mult. by 12, $6x+42-8x+28=3x-180$
Trans. and unit. terms, $5x=250$ Div. $x=50$.

Ex. 18. Reduce $\frac{6x-1}{7}-\frac{3x-8}{5}=\frac{16-x}{2}-1+\frac{4x+1}{5}-x$.

Mult. by 70,
$60x-10-42x+112=560-35x-70+56x+14-70x$
Trans. $60x-42x+35x-56x+70x=560-70+14-112+10$
Unit. terms, $67x=402$ Div. $x=6$.

Ex. 19. Reduce $\frac{2x-3}{4}-\frac{4x-9}{5x-6}=\frac{x-1}{2}-\frac{7}{12}$.

Mult. by $60x-72$,
$30x^2-81x+54-48x+108=30x^2-66x+36-35x+42$
Trans. $-81x-48x+66x+35x=36+42-108-54$
Unit. terms, $28x=84$ Div. $x=3$.

Ex. 20. Reduce $(a-x)\ (b+x)-b(a-c)+x^2=\frac{a^2c}{b}$.

Mult. by $b$, $(a-x)\ (b^2+bx)-b^2(a-c)+bx^2=a^2c$
Or, $ab^2-b^2x+abx-bx^2-ab^2+b^2c+bx^2=a^2c$
Trans. and unit. terms, $abx-b^2x=a^2c-b^2c$

Div. by $(ab-b^2)$ $x=\frac{c(a^2-b^2)}{ab-b^2}=\frac{c(a^2-b^2)}{b(a-b)}$

Div. num. and denom. by $(a-b)$, (ART. 111), $x=\frac{c}{b}(a+b)$.

Ex. 21. Reduce $\frac{7-x}{6}:5::\frac{5x-4}{3}:12.$

Mult. extremes and means, $\frac{84-12x}{6}=\frac{25x-20}{3}$

Clearing of fractions, $84-12x=50x-40$

Trans. and uniting terms, $62x=124$, and $x=2$.

Note.—It is to be observed in the solution of these examples, that the equations are cleared of fractions by multiplying both sides by the least common multiple of the denominators.

## Problems.

Art. 199. Prob. 7. Let the number be denoted by $x$.

By the conditions, $2x=\frac{2x}{6}+45.$

Mult. by 6, $12x=2x+270$

Transposing, $10x=270$ Div. $x=27$.

Prob. 8. Let $x$ denote the price per bushel.

Then $14x$ = the amount received for 14 bushels.

$22x$ = " " 22 "

By the conditions, $22x-14x=56$

Or, $8x=56$, and $x=7$.

Prob. 9. Let $x$ denote the required number.

By the conditions, $3x-25=38-\frac{x}{2}$

Mult. by 2, $6x-50=76-x$

Transposing, $7x=126$, and $x=18$.

Prob. 10. Let $x$ denote the num. of gals. in the 4th cask.

Then $2x$, $4x$, and $12x$ will equal the number of gallons in the third, second, and first respectively.

By the conditions, $12x+4x+2x+x=209$

Uniting terms, $19x=209$, and $x=11$

Therefore $2x=22$=gallons in the 3d, $4x=44$=gallons in the 2d, and $12x=132$=gallons in the 1st.

**Prob.** 11. Let $x=$ the greater, then the smaller will $=75-x$.

By the conditions, $\frac{x}{5}-\frac{75-x}{10}=6$

Mult. by 10, $2x-75+x=60$

Trans. and unit. terms, $3x=135$

Therefore $x=45=$ the greater, $75-x=30=$ the smaller part.

**Prob.** 12. Let $x=$ the first piece of metal.

Then $x+5=$ the second, and $x+5+7=$ the third.

Also, $x+x+5+x+5+7=62$

Trans. and unit. terms, $3x=45$, and $x=15$.

Therefore the three pieces weigh 15, 20, 27 oz. respectively.

**Prob.** 13. Let $x=$ the first number.

Then $x+m=$ the second, $x+m+n=$ the third.

By supposition, $x+x+m+x+m+n=a$

Trans. and unit. terms, $3x=a-2m-n$

Hence, $x=\frac{a-2m-n}{3}=$ the first.

$\frac{a-2m-n}{3}+m=\frac{a+m-n}{3}=$ the second,

$\frac{a+m-n}{3}+n=\frac{a+m+2n}{3}=$ the third.

**Prob.** 14. Let $x$ denote the number.

By the conditions, $\frac{x}{2}+\frac{x}{3}+\frac{x}{4}=x+6$

Mult. by 12, $6x+4x+3x=12x+72$

Trans. and unit. terms, $x=72$.

**Prob.** 15. Let $x=$ the sum received by each.

By the conditions, $x+12=(x-3)\times 4$

Transposing, $3x=24$, and $x=8$.

**Prob.** 16. Let $x$ denote the number.

By supposition, $(x-b)m=x+a$

Transposing, $mx-x=a+bm$, Div. $x=\frac{a+bm}{m-1}$.

**Prob.** 17. Let $x$ = the cavalry.

By the conditions, $x+4x+7x=1800$

Unit. terms, $12x=1800$, and $x=150$.

Hence $4x=600$= the artillery, $7x=1050$= the infantry.

**Prob.** 18. Let $x$ = the third part, then $nx$ = the second, $mx$ = the first.

By supposition, $x+mx+nx=a$

Div. by coeff. of $x$, $x=\dfrac{a}{m+n+1}$

Hence $\dfrac{am}{m+n+1}$ = the first, and $\dfrac{an}{m+n+1}$ = the second.

**Prob.** 19. Let $4x$=A's portion, then $7x$=B's, and $11x$=C's.

By the conditions, $4x+7x+11x=660$

Unit. terms, $22x=660$, and $x=30$

Hence $4x=120$=A's portion, $7x=210$=B's, $11x=330$=C's.

**Prob.** 20. Let $x$ = the weight of the flour.

Then $\frac{x}{5}-2$= weight of rice, and $\frac{6x-10}{40}$ = weight of water.

By the conditions, $x+\frac{x}{5}-2+\frac{6x-10}{40}=18$

Mult. by 40, $40x+8x-80+6x-10=720$

Trans. and unit. terms, $54x=810$, and $x=15$

Hence $\frac{x}{5}-2=1$=weight of rice, $\frac{6x-10}{40}=2$=weight of water.

**Art.** 200. **Prob.** 21. Let $2x$, $3x$, and $5x$ represent the pieces.

Then $2x+3x+5x=10x$= the whole quantity.

Therefore by the supposition, $2x-12+3x-12+5x-12=7x$

Trans. and uniting terms, $3x=36$, 2d piece;

And $2x=24$, 1st piece; $5x=60$, 3d piece.

**Prob.** 22. Let $x$ = the less, $x+21$ = the greater.

By the conditions, $2x+21-9=3x$

Transposing, $x=12$, the less, $x+21=33$, the greater num.

PROB. 23. Let $x=$ number of men.

If $x$ men work, each man does $\frac{1}{x}$ part of the whole work.

If $x+5$ " " " " " $\frac{1}{x+5}$ " " " "

The amount of work done by each is proportioned to the time occupied.

Therefore, $\frac{1}{x}:\frac{1}{x+5}::6:4$

Mult. extremes and means, $\frac{4}{x}=\frac{6}{x+5}$

Clearing of fractions, $4x+20=6x$

Transposing, $2x=20$, and $x=10$.

PROB. 24. Let $x=$ number of inhabitants in A.

Then $\frac{3}{2}x=$num. of inhab. in B, and $\frac{7}{5}\times\frac{3}{2}x=$num. of inhab. in C.

By the conditions, $x+\frac{3}{2}x+\frac{21}{10}x=4600$

Mult. by 10, $10x+15x+21x=46000$

Uniting terms, $46x=46000$, and $x=1000$

Therefore, $\frac{3}{2}x=1500$, num. in B, $\frac{21}{10}x=2100$, num. in C.

PROB. 25. Let $x=$ the whole sum.

$\frac{x}{2}-1000=$ A's share, $\frac{x}{3}+200=$B's, $\frac{x}{4}+300=$ C's.

And $\frac{x}{2}-1000+\frac{x}{3}+200+\frac{x}{4}+300=x$

Mult. by 12, $6x-12000+4x+2400+3x+3600=12x$

Trans. and unit. terms, $x=6000$

$\frac{x}{2}-1000=2000$, A's share, $\frac{x}{3}+200=2200$, B's share,

and $\frac{x}{4}+300=1800$, C's share.

**Prob.** 26. Let $x=$ the original capital.

Then $\frac{2}{3}x+400=$2d cap. $\frac{3}{4}\left(\frac{2}{3}x+400\right)+200=\frac{x}{2}+500=$3d cap.

By supposition, $$\frac{x}{2}+500-\frac{1}{5}\left(\frac{x}{2}+500\right)=1600$$

Or, $$\frac{4}{5}\left(\frac{x}{2}+500\right)=1600$$

And $$\frac{2x}{5}+400=1600$$

Mult. by 5, $$2x+2000=8000$$

Hence, $$2x=6000,\text{ and } x=3000$$

**Prob.** 27. Let $x-7=$ the 1st part.

Then $x+5=$2d, $\frac{x}{4}=$3d, and $3x=$4th.

Also $$x-7+x+5+\frac{x}{4}+3x=103$$

Unit. terms and mult. by 4, $21x=420$, and $x=20$

Hence $x-7=13$, 1st part; $x+5=25$, 2d;

$\frac{x}{4}=5$, 3d; and $3x=60$, 4th.

**Prob.** 28. Let $x=$B, $\frac{3}{4}x=$A.

By the conditions, $$x-\left(\frac{2}{3}x+12\right)=\frac{2}{5}\left(\frac{3}{4}x-\frac{3}{8}x+3\right)$$

By reduction, $$\frac{x}{3}-12=\frac{3}{20}x+\frac{6}{5}$$

Mult. by 60, $$20x-720=9x+72$$

Transposing, $11x=792$, $x=72=$B, $\frac{3}{4}x=54=$A.

**Prob.** 29. Let $x=$ the fraction of a pound of the 12 cent sugar, contained in 1 pound of the mixture; then $(1-x)$ will denote the portion of the 7 cent sugar in 1 pound of the mixture.

Now $12\times x$ = the value of the portion $x$
And $7(1-x)=$ “ “ “ $1-x$
Therefore by the supposition, $12x+7(1-x)=10$
Trans. and unit. terms, $3x=5$, $x=\frac{3}{5}$, and $1-x=\frac{2}{5}$

Hence 1 pound of the mixture must contain $\frac{3}{5}$ of a pound of the sugar worth 12 cents a pound, and $\frac{2}{5}$ of a pound of the sugar worth 7 cents a pound.

**Prob. 30.** (This is the preceding problem generalized.) Let $x$ = the portion of the $m$ cent sugar in 1 lb. of the mixture.
Then $(1-x)=$ “ “ $n$ “ “ “ “ “
Now $mx$ = the worth of the portion $x$
And $n(1-x)=$ “ “ “ $(1-x)$
Therefore by the supposition, $mx+n(1-x)=r$
Transposing, $(m-n)x=r-n$, $x=\frac{r-n}{m-n}$
And $1-x=1-\frac{r-n}{m-n}=\frac{m-r}{m-n}$

Therefore each pound of the mixture must contain $\frac{r-n}{m-n}$ of a pound of the $m$ cent sugar, and $\frac{m-r}{m-n}$ of a pound of the $n$ cent sugar.

**Prob. 31.** Let $3x$ = the larger bucket, and $2x$ = the smaller. Then since the smaller was emptied 36 times, while the larger was emptied $\frac{3}{4}$ as often, or 27 times,
We have $72x+27\times 3x=765$
Uniting terms, $153x=765$, and $x=5$
Therefore $3x=15$, the larger, and $2x=10$, the smaller bucket.

### PROBLEM 32.

Let $x$ = number of days the flour would last the man alone.
$\frac{1}{x}$ = the portion he would consume in a day.

$\frac{1}{60}=$ the portion his family consume in a day when he is absent.

$\frac{1}{50}=$ what the man and his family together consume in a day.

Therefore $\frac{1}{x}=\frac{1}{50}-\frac{1}{60}$

Mult. by $300x$, $300=6x-5x$

Or, $x=300$ days, the time required.

### Problem 33.

Let $x=$ number of days the flour would last the man alone.

$\frac{1}{x}=$ the portion he would consume in a day.

$\frac{1}{m+n}=$ the portion his family consume in a day when he is absent.

$\frac{1}{m}=$ what the man and his family together consume in a day.

Therefore $\frac{1}{m}-\frac{1}{m+n}=\frac{1}{x}$

Clear. of fractions $mx+nx-mx=m(m+n)$

And $x=\frac{m(m+n)}{n}$ days.

Prob. 34. Let $x=$ the number of gallons of each kind.

The merchant paid $4x$ for the one, $7x$ for the other, and received $6\times 2x$ for the mixture.

By supposition, $12x-18=4x+7x$

By reduction, $x=18$, num. of gals. of each.

Prob. 35. Let $x=$ the number of gallons of each kind.

$mx$ is paid for the one kind, $nx$ for the other, and $2rx$ is received for the mixture.

By supposition, $2rx-a=mx+nx$

By reduction, $x=\frac{a}{2r-m-n}$ gallons.

**Prob. 36.** Let $x=$ the number.

By the conditions, $x+2:x+5::x+5:x+11$

Mult. extremes and means, $x^2+13x+22=x^2+10x+25$

Transposing, $3x=3$, and $x=1$.

**Prob. 37.** Let $x=$ the number.

By the conditions, $x+a:x+b::x+b:x+c$

Mult. extremes and means, $x^2+ax+cx+ac=x^2+2bx+b^2$

By reduction, $ax+cx-2bx=b^2-ac$, and $x=\dfrac{b^2-ac}{a+c-2b}$.

**Prob. 38.** Let $x=$ A's share.

Then $x-300=$ B's, and $x+200=$ C's.

By the conditions, $x+x-300:x-300+x+200::7:9$

Or, $2x-300:2x-100::7:9$

Mult. extremes and means, $18x-2700=14x-700$

Transposing, $4x=2000$, and $x=500$, A's share

And $x-300=200$, B's, $x+200=700$, C's.

**Prob. 39.** Let $x=$ A's share.

Then $x-m=$ B's, and $x+n=$ C's.

By the conditions, $2x-m:2x-m+n::p:q$

Mult. extremes and means, $2qx-mq=2px-mp+np$

By reduction, $2qx-2px=p(n-m)+qm$

Div. by $2(q-p)$ $x=\dfrac{p(n-m)+qm}{2(q-p)}$

And $x-m=\dfrac{p(n+m)-qm}{2(q-p)}$, and $x+n=\dfrac{q(2n+m)-p(n+m)}{2(q-p)}$.

**Prob. 40.** Let $x=$ the number of acres of wheat.

Then $26-x=$ acres of rye, $(26-x)\times 1=$ wages for reaping the rye, and $x\times 1.25=\frac{5}{4}x=$ wages for reaping the wheat.

By the supposition, $26-x+\frac{5}{4}x=30$

Uniting terms, $\frac{1}{4}x=4$, and $x=16$

Therefore, $16\times 1.25=\$20$, wages for reaping the wheat.

PROB. 41. Let $x$ and $15-x$ be the two parts.

By the conditions, $x^2-(225-30x+x^2)=105$

Trans. and unit. terms, $30x=330$, and $x=11$, 1st part.

And $15-x=4$, 2d part.

PROB. 42. Let $x$ and $a-x$ be the parts.

By the conditions, $x^2-(a^2-2ax+x^2)=d$

By reduction, $2ax=a^2+d$, and $x=\frac{a^2+d}{2a}$

Also $a-x=a-\frac{a^2+d}{2a}=\frac{a^2-d}{2a}$

PROB. 43. Let $x=$ the original sum. Then $x-432=$ remainder after paying creditors; $x-432-\frac{1}{3}(x-432)=\frac{2}{3}x-288=$ remainder after lending his friend.

Therefore $\frac{2}{3}x-288-\frac{1}{4}\left(\frac{2}{3}x-288\right)=\frac{x}{5}$

Or, $\frac{2}{3}x-288-\frac{1}{6}x+72=\frac{x}{5}$

Uniting terms, $\frac{x}{2}-216=\frac{x}{5}$

Mult. by 10, $5x-2160=2x$, $3x=2160$, and $x=720$.

PROB. 44. The dial plate of a clock is divided into 60 equal spaces. When the clock strikes two, the distance between the hands is 10 of these spaces. Now let $x=$ the distance passed over by the hour hand after two o'clock before the minute hand overtakes it; then $x+10$ will equal the distance passed over by the minute hand during the same interval of time, and sinco the minute hand moves 12 times faster than the hour hand

We have the equation $12x=10+x$, and $x=\frac{10}{11}$

Therefore the time required is $10\frac{10}{11}$ minutes past twc

Prob. 45. Let $x=$ number of gallons; then $\frac{150}{x}=$ price per gallon, $\left(\frac{2}{3}x-5\right)=$ quantity first sold, at a profit of 50 per cent.; therefore at $\frac{225}{x}$ per gallon: and the amount received at the first sale is $\left(\frac{2}{3}x-5\right)\frac{225}{x}=150-\frac{1125}{x}$.

$\left(\frac{1}{3}x+5\right)=$ quantity sold last, at a profit of 125 per cent.; therefore at $\frac{337.5}{x}$ per gallon; and the amount received at the second sale is $\left(\frac{1}{3}x+5\right)\frac{337.5}{x}=112.5+\frac{1687.5}{x}$.

By the supposition, $150-\frac{1125}{x}+112.5+\frac{1687.5}{x}=150\times 1.80$

Clear. of frac. $150x-1125+112.5x+1687.5=270x$

Trans. and unit. terms, $7.5x=562.5$, and $x=75$.

Prob. 46. Let $x=$ the whole number of ships.

Then $\frac{x}{5}=$ num. taken; $\frac{x}{5}-1=$ num. burnt; $\frac{x}{5}+2=$ num. sunk.

By the conditions, $x=\frac{x}{5}+\frac{x}{5}-1+\frac{x}{5}+2+13$

Mult. by 5, $5x=x+x-5+x+10+65$

Uniting terms, $2x=70$, and $x=35$.

Prob. 47. Let $x=$ the quantity of wheat.

When wheat is worth a dollar a bushel, the value of the wheat is $1\times x$; and the whole rent in this case is $80+x$; which is 8 dollars per acre, and hence the number of acres is $\frac{80+x}{8}$.

At \$1.25 per bushel, the value of the wheat is $x\times 1.25=\frac{5}{4}x$;

and the whole rent will be $80+\frac{5}{4}x$; which brings 9 dollars per acre, and hence the number of acres is $\frac{320+5x}{4\times 9}$.

Equating the two expressions for the acres, $\frac{80+x}{8}=\frac{320+5x}{4\times 9}$

Clearing of fractions, $720+9x=640+10x$

Transposing, $x=80$

Subs. in number of acres, $\frac{80+x}{8}=\frac{160}{8}=20$ acres.

Prob. 48. Let $x=$ the quantity of wheat.

When wheat is worth $m$ dollars a bushel, the value of the wheat is $mx$; and the whole rent in this case is $a+mx$; which brings in $p$ dollars per acre, and hence the number of acres is $\frac{a+mx}{p}$.

At $n$ dollars per bushel, the value of the wheat is $nx$; and the whole rent will be $a+nx$; which is $q$ dollars per acre, and hence the number of acres is $\frac{a+nx}{q}$.

Equating the two expressions for the acres, $\frac{a+mx}{p}=\frac{a+nx}{q}$

Clearing of fractions, $aq+mqx=ap+npx$

Transposing, $(np-mq)x=aq-ap$, and $x=\frac{aq-ap}{np-mq}$.

Subs. this value in the number of acres, we shall have

$$\frac{a+nx}{q}=\frac{a}{q}+\frac{anq-anp}{q(np-mq)}=\frac{anp-amq+anq-anp}{q(np-mq)}=\frac{a(n-m)}{np-mq}\text{ acres.}$$

Prob. 49. Let $x=$ the distance B must travel; then $x-15$ will equal the distance passed over by A in the same time. And since B goes five miles while A goes four, the whole distances must be in the ratio of these numbers.

Therefore, $x:x-15::5:4$

Mult. extremes and means, $4x=5x-75$, and $x=75$.

Prob. 50. Let $x =$ the distance B must travel; then $x-d$ will equal the distance passed over by A in the same time. And since B goes $m$ miles while A goes $n$, the whole distances must be in the ratio of these numbers.

Therefore, $x : x-d :: m : n$

Mult. extremes and means, $nx=mx-md$

And $x=\dfrac{md}{m-n}$.

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## Simple Equations containing Two Unknown Quantities.

Prob. 2. Let $x =$ the greater number, and $y =$ the less.

By the 1st condition, $x-y=a$ (1)

" 2d " $x=my$ (2)

Trans. $y$ in (1), $x=a+y$ (3)

Equating (2) and (3), $my=a+y$

Or, $my-y=a$, and $y=\dfrac{a}{m-1}$

Subst. this value of $y$ in (2), $x=\dfrac{am}{m-1}$.

Prob. 3. $\left.\begin{array}{ll}\text{Given} & ax-by=c \\ \text{And} & x-y=d\end{array}\right\}\begin{array}{ll}\text{Trans. and div.} & -y=\dfrac{c-ax}{b} \\ \text{Trans.} & -y=d-x\end{array}$

Equating these values of $-y$, $\dfrac{c-ax}{b}=d-x$

Mult. by $b$, $c-ax=bd-bx$

Transposing, $bx-ax=bd-c$, and div. $x=\dfrac{bd-c}{b-a}$.

Prob. 5. Let $x =$ the numerator, and $y =$ the denominator.

By the conditions, $\left.\begin{array}{l}\dfrac{x+2}{y}=\dfrac{2}{3} \\ \dfrac{x}{y+1}=\dfrac{1}{2}\end{array}\right\}$ Clearing of fractions, $\left\{\begin{array}{lr}3x+6=2y & (1) \\ 2x=y+1 & (2)\end{array}\right.$

Find. the value of $y$ in (1) and subst. in (2), $2x=\frac{3x+6}{2}+1$

Mult. by 2, and trans. $4x-3x=6+2$, and $x=8$

Subst. this value of $x$ in (2), $y+1=16$, and $y=15$

Therefore the fraction is $\frac{8}{15}$.

**Prob. 6.** Let $x=$ the greater number, and $y=$ the less.

By the conditions, $x+y:x-y::9:4$ (1)

And $x-2y=3$ (2)

Mult. extremes & means of (1), $4x+4y=9x-9y$, or $5x=13y$ (3)

But (2) gives $x=3+2y$; and substituting this value in (3)

We have $15+10y=13y$, or $3y=15$, and $y=5$.

$$\left.\begin{aligned}&\textbf{Prob. 11.}\ \text{Given}\ \frac{x}{5}+\frac{y}{6}=2\\&\text{And}\ \frac{x}{3}+\frac{y}{4}=\frac{19}{6}\end{aligned}\right\}\ \begin{matrix}\text{Clearing}\\\text{of}\\\text{fractions.}\end{matrix}\ \left\{\begin{aligned}6x+5y&=60\quad(1)\\4x+3y&=38\quad(2)\end{aligned}\right.$$

Mult. (1) by 2, $12x+10y=120$ (3)

" (2) " 3, $12x+9y=114$ (4)

Subtracting (4) from (3), $y=6$

Subst. this value in (1), $6x+30=60$, or $6x=30$, and $x=5$.

**Prob. 12.** Given $(x+1)(y+2)=(x-1)(y-2)+26$ (1)

And $5x-8y=1$ (2)

Mult. factors and unit. terms of (1), $4x+2y=26$ (3)

Mult. (3) by 4 $16x+8y=104$ (4)

Adding (2) and (4), $21x=105$, and $x=5$

Subst. this value in (2), $25-8y=1$, or $8y=24$, and $y=3$.

**Prob. 13.** Given $\frac{x-4}{3}+\frac{y-x}{2}+\frac{10-y}{12}=\frac{19}{12}$

And $\frac{x+y}{6}+\frac{x+1}{2}-\frac{y-1}{3}=3$

Clearing of fractions, $\left\{\begin{aligned}4x-16+6y-6x+10-y&=19\\x+y+3x+3-2y+2&=18\end{aligned}\right.$

Uniting terms, $\begin{cases} 5y-2x=25 & (1) \\ 4x-y=13 & (2) \end{cases}$

Mult. (1) by 2 and add. it to (2), $-4x+10y=50$

$9y=63$, and $y=7$

Subst. this value in (2), $4x-7=13$, or $4x=20$, and $x=5$.

### PROBLEM 15.

Given $\frac{a}{x}-\frac{b}{y}=m$ (1) Mult. (1) by $c$, $\frac{ac}{x}-\frac{bc}{y}=cm$ (3)

And $\frac{c}{x}-\frac{d}{y}=n$ (2) " (2) " $a$, $\frac{ac}{x}-\frac{ad}{y}=an$ (4)

Subtract. (3) from (4), $\frac{bc}{y}-\frac{ad}{y}=an-cm$

Or, $bc-ad=y(an-cm)$, and $y=\frac{bc-ad}{an-cm}$.

Again, mult. (1) by $d$, $\frac{ad}{x}-\frac{bd}{y}=dm$ (5)

And " (2) " $b$, $\frac{bc}{x}-\frac{bd}{y}=bn$ (6)

Subtract. (5) from (6), $\frac{bc}{x}-\frac{ad}{x}=bn-dm$

Or, $bc-ad=x(bn-dm)$, and $x=\frac{bc-ad}{bn-dm}$.

PROB. 16. Given $\frac{10x+y}{x+1}=9$, and $\frac{10x+y}{y-1}=\frac{9}{2}$.

Clear. of fract. $\begin{cases} 10x+y=9x+9, \text{ or } x+y=9 & (1) \\ 20x+2y=9y-9, \text{ or } 20x+7y=-9 & (2) \end{cases}$

Mult. (1) by 7 and add. it to (2), $7x+7y=63$

$27x=54$, and $x=2$

Subst. this value in (1), $2+y=9$, or $y=7$, and the num. is 27.

PROB. 17. Let $10x+y=$ the number required.

By the conditions, $\frac{10x+y}{x+m}=a$ (1), and $\frac{10x+y}{y-n}=b$ (2)

Clear. (1) of fract. $10x+y=ax+am$, or $(10-a)x+y=am$ (3)

" (2) " $10x+y=by-bn$, or $10x+(1-b)y=-bn$ (4)

Mult. (3) by $(1-b)$,

$$10x-ax-10bx+abx+(1-b)y=am-abm \qquad (5)$$

Subtract. (5) from (4),

$$x(a+10b-ab)=abm-am-bn, \text{ or } x=\frac{abm-am-bn}{a+10b-ab}$$

Again, mult. (3) by 10, $10(10-a)x+10y=10am$ (6)

And " (4) " $(10-a)$,

$$10(10-a)x+y(10-a-10b+ab)=abn-10bn \qquad (7)$$

Subtract. (7) from (6),

$$y(a+10b-ab)=10bn-abn+10am, \text{ or } y=\frac{10bn-abn+10am}{a+10b-ab}$$

Therefore,

$$10x+y=\frac{10abm-10am-10bn+10bn-abn+10am}{a+10b-ab}=$$

$$\frac{10abm-abn}{a+10b-ab}=\frac{ab(10m-n)}{a+10b-ab}=\text{the number required.}$$

PROB. 18. Let $x=$ the numerator, and $y=$ the denominator.

$$\left.\begin{array}{ll}\text{Then} & \frac{2x}{y-2}=\frac{7}{5} \\ \text{And} & \frac{x-3}{2y}=\frac{1}{6}\end{array}\right\}\text{Clear. of fract.}\left\{\begin{array}{ll}10x=7y-14 & (1) \\ 6x-18=2y & (2)\end{array}\right.$$

Mult. (1) by 2, $20x=14y-28$ (3)

" (2) " 7, $42x-126=14y$ (4)

Subtract. (3) from (4), $22x=154$, or $x=7$

Subst. this value in (2), $2y=42-18=24$, or $y=12$

And the fraction is $\frac{7}{12}$.

PROB. 19. Let $x=$ one number, and $y=$ the other.

By the conditions, $\frac{x+y}{x}=\frac{9}{5}$ (1), and $x-y=\frac{x}{2}-6$ (2)

Mult. (1) by $x$, $x+y=\frac{9x}{5}$ (3)

Adding (2) and (3), $2x=\frac{9x}{5}+\frac{x}{2}-6$

Or, $20x=18x+5x-60$, $3x=60$, and $x=20$

Subst. this value in (2), $20-y=10-6$, or $y=16$.

Prob. 20. Let $x=$ one number, and $y=$ the other.

By the conditions, $\frac{x+y}{x}=a$ (1), and $x-y=\frac{x}{2}-b$ (2)

Mult. (1) by $x$, $x+y=ax$ (3)

Adding (2) and (3), $2x=ax+\frac{x}{2}-b$

Or, $4x=2ax+x-2b$, $(2a-3)x=2b$, and $x=\frac{2b}{2a-3}$

Subst. this value in (2), $\frac{2b}{2a-3}-y=\frac{b}{2a-3}-b$

Or, $y=\frac{b}{2a-3}-b=\frac{2b(a-1)}{2a-3}$.

Prob. 21. Let $x=$ A's money, and $y=$ B's.

By the conditions, $x+5=\frac{y-5}{2}$ (1), and $y+10=5x-50$ (2)

Mult. (1) by 2 and trans. $-y+10=-2x-5$ (3)

Adding (2) and (3), $20=3x-55$, $3x=75$, and $x=25$

Subst. this value in (2), $y+10=125-50$, or $y=65$

Prob. 22. Let $x=$ the barrels of wheat, $y=$ barrels of rye. Then $5x+3y=$ the money paid; $6x+6y=$ the money received; and $\frac{33\frac{1}{3}(5x+3y)}{100}=\frac{5x+3y}{3}=$ the gain.

By the conditions, $x+y=40$, or $x=40-y$ (1)

And $6x+6y=5x+3y+\frac{5x+3y}{3}$ (2)

Clear. (2) of fractions, $18x+18y=15x+9y+5x+3y$

Uniting terms, $6y=2x$, and $y=\frac{x}{3}$

Subst. this value in (1), $x=40-\frac{x}{3}$, $3x=120-x$, and $x=30$.

**Prob. 23.** Let $x=$ the wheat, and $y=$ the rye.

Then $mx+ny=$ the money paid; $r(x+y)=$ the money received; and $\frac{mpx+npy}{100}=$ the gain.

By the conditions, $x+y=a$, or $x=a-y$ (1)

And $mx+ny+\frac{pmx+pny}{100}=r(x+y)$ (2)

Clear. (2) of fract. $100mx+100ny+pmx+pny=100rx+100ry$

Transposing, $100ry-100ny-pny=100mx-100rx+pmx$

Dividing by coefficient of $y$, $y=\frac{100mx-100rx+pmx}{100r-100n-pn}$

Subst. this value in (1), $x=a-\frac{100mx-100rx+pmx}{100r-100n-pn}$

Clearing of fractions, $100rx-100nx-pnx=$
$a(100r-100n-pn)-100mx+100rx-pmx$

Or, $x(100m-100n+pm-pn)=a(100r-100n-pn)$

And $x=\frac{a(100r-100n-pn)}{(m-n)\,(100+p)}=$ the amount of wheat.

**Prob. 24.** Let $x=$ the time in which A alone could do the work; $y=$ the time in which B alone could do it: then $\frac{1}{x}$ and $\frac{1}{y}$ will respectively equal the amount of work done by A and B in one day: also, $\frac{4}{x}+\frac{4}{y}=$ the amount of work completed when A is called off, and $\frac{20}{y}$ equals the remaining portion of the work which B performs alone.

Then $\frac{12}{x}+\frac{12}{y}=1$, the whole work.

And $\frac{4}{x}+\frac{4}{y}+\frac{20}{y}=1$, " "

Clearing of fractions, $12y+12x=xy$ (1)

" " $4y+24x=xy$ (2)

Equating these values of $xy$, $12y+12x=4y+24x$, or $y=\frac{3}{2}x$

Subst. this value of $y$ in (1), $\frac{36x}{2}+12x=\frac{3}{2}x^2$

Dividing by $x$, and reducing, $\frac{3}{2}x=30$, and $x=20$.

PROB. 25. Let $x=$ the time in which A alone could do the work, and $y=$ that in which B alone could do it: then $\frac{1}{x}$ and $\frac{1}{y}$ will respectively equal the amount of work done by A and B in one day: also, $\frac{n}{x}+\frac{n}{y}=$ the amount of work completed when A is called off, and $\frac{r}{y}=$ the remaining portion of the work which B performs alone.

Then $\frac{m}{x}+\frac{m}{y}=1$, the whole work.

And $\frac{n}{x}+\frac{n}{y}+\frac{r}{y}=1$, " "

Clearing of fractions, $my+mx=xy$ (1)

" " $ny+(n+r)x=xy$ (2)

Equating these values of $xy$, $my+mx=ny+(n+r)x$

Transposing and div. by coeff. of $y$, $y=\frac{(n+r-m)x}{m-n}$

Subst. this value in (1), $\frac{mx(n+r-m)}{m-n}+mx=\frac{x^2(n+r-m)}{m-n}$

Mult. by $(m-n)$, and div. by $x$,

$$mn+mr-m^2+m^2-mn=x(n+r-m)$$

And $x=\frac{mr}{n+r-m}$.

## Simple Equations containing Three or More Unknown Quantities.

PROB. 28. Given $\begin{cases} x+y-z=a & (1) \\ x+z-y=b & (2) \\ y+z-x=c & (3) \end{cases}$

Adding (1) and (2), $2x=a+b$, and $x=\frac{a+b}{2}$.

" (1) " (3), $2y=a+c$, and $y=\frac{a+c}{2}$

" (2) " (3), $2z=b+c$, and $z=\frac{b+c}{2}$.

PROB. 29. Let $x$, $y$, and $z$ be the three parts.

Then the conditions furnish, $\begin{cases} x+y+z=46 & (1) \\ x+10 : y+10 :: 5 : 9 & (2) \\ y-3 : z-3 :: 2 : 3 & (3) \end{cases}$

Mult. extremes and means of (2) and (3), $\begin{cases} 9x+90=5y+50, \text{ or } 9x-5y=-40 & (4) \\ 3y-9=2z-6, \text{ or } 3y-2z=3 & (5) \end{cases}$

Mult. (1) by 2 and add. to (5), $2x+5y=95$ (6)

Adding (4) and (6), $11x=55$, and $x=5$

Subst. this value in (4), $45-5y=-40$, or $5y=85$, and $y=17$

Subst. the values of $x$ and $y$ in (1), $5+17+z=46$, or $z=24$.

PROB. 30. Let $x=$ A's money, $y=$ B's, and $z=$ C's.

By the 1st cond. $\left.\begin{array}{l} x+10=2y-20 \\ y+20=3z-60 \\ z+6=5x-30 \end{array}\right\}$ Trans. $\begin{cases} x-2y=-30 & (1) \\ y-3z=-80 & (2) \\ z-5x=-36 & (3) \end{cases}$

" 2d "

" 3d "

Mult. (2) by 2, and add. to (1), $x-6z=-190$ (4)

Mult. (3) by 6, and add. to (4), $-29x=-406$, or $x=14$

Subst. this value of $x$ in (1), $14-2y=-30$, $2y=44$, $y=22$

Subst. the same in (3), $z-70=-36$, or $z=34$

### PROBLEM 31.

$$\text{Given}\left\{\begin{array}{l} x+\frac{y}{2}+\frac{z}{3}=18 \\ y+\frac{z}{2}+\frac{x}{3}=23 \\ z+\frac{x}{2}+\frac{y}{2}=25 \end{array}\right\}\begin{array}{c}\text{Clearing}\\ \text{of}\\ \text{fractions,}\end{array}\left\{\begin{array}{ll} 6x+3y+2z=108 & (1) \\ 6y+3z+2x=138 & (2) \\ 6z+3x+2y=150 & (3) \end{array}\right.$$

Mult. (2) by 3, and subtract. (1) from it, $15y+7z=306$ (4)
Mult. (3) by 2, and subtract. (1) from it, $y+10z=192$ (5)
Mult. (5) by 15, and subtract. (4) from it, $143z=2574$, or $z=18$
Subst. this value of $z$ in (5), $y+180=192$, or $y=12$
Subst. these values of $y$ and $z$ in (1),

$$6x+36+36=108, \quad 6x=36, \quad x=6.$$

PROB. 32. Given

$$\left\{\begin{array}{ll} \frac{1}{x}+\frac{1}{y}=a & (1) \\ \frac{1}{x}+\frac{1}{z}=b & (2) \\ \frac{1}{y}+\frac{1}{z}=c & (3) \end{array}\right.$$

Subtract. (2) from (1), $\frac{1}{y}-\frac{1}{z}=a-b$ (4)

Adding (4) to (3) $\frac{2}{y}=a-b+c$, or $y=\frac{2}{a-b+c}$

In like manner by subtract. (3) from (1), and adding the result to (2), we obtain $x=\frac{2}{a+b-c}$

Also, by subtracting (1) from (3), and adding the result to (2), we obtain $z=\frac{2}{b+c-a}$.

PROB. 33. Let $x$, $y$, and $z$ be the three digits.

By the 1st condition, $y=\frac{x+z}{2}$

" 2d " $\frac{100x+10y+z}{x+y+z}=26$

" 3d " $100x+10y+z+198=100z+10y+x$

Clear. of fract. and unit. terms, $\left\{\begin{aligned} 2y-x-z&=0 \qquad (1)\\ 74x-16y-25z&=0 \qquad (2)\\ 99x-99z&=-198 \quad (3)\end{aligned}\right.$

Dividing (3) by 33, $x-z=-2$, or $x=z-2$

Subst. this value of $x$ in (1), $2y-2z+2=0$, or $y=z-1$

Subst. these values of $x$ and $y$ in (2),

$$74(z-2)-16(z-1)-25z=0$$

Or, $74z-148-16z+16-25z=0$, $33z=132$, and $z=4$

Therefore $y=4-1=3$, $x=4-2=2$, and the number is 234.

**Prob. 35.** Given $\left\{\begin{aligned} 9u-4x+10y+3z&=9 \qquad (1)\\ 9u+3x+6y-2z&=6 \qquad (2)\\ -6u-5x+4y+5z&=5 \qquad (3)\\ -3u-3x+8y+2z&=3 \qquad (4)\end{aligned}\right.$

Subtracting (2) from (1), $-7x+4y+5z=3$ (5)

Mult. (4) by 3, and add. it to (1), $-13x+34y+9z=18$ (6)

Mult. (4) by 2, and add. it to (3), $x-12y+z=-1$ (7)

Mult. (7) by 5, and subtract. it from (5),

$$-12x+64y=8, \text{ or } -3x+16y=2 \qquad (8)$$

Mult. (7) by 9, and subtract. it from (6),

$$-22x+142y=27 \qquad (9)$$

Mult. (8) by 22, $-66x+352y=44$

Mult. (9) by 3, $-66x+426y=81$

Taking the diff. of the last two results, $74y=37$, or $y=\frac{1}{2}$

Subst. this value in (8), $-3x+8=2$, or $x=2$

Subst. the values of $x$ and $y$ in (7), $2-6+z=-1$, or $z=3$

Subst. the values of $x$, $y$, and $z$ in (1), $9u-8+5+9=9$, or $u=\frac{1}{3}$.

**Prob. 36.** Let $x$, $y$, $u$, and $z$ denote the parts.

By the conditions, $\left\{\begin{aligned} x+y+z+u&=18 \qquad (1)\\ x+\frac{y+z+u}{2}&=10 \qquad (2)\\ y+\frac{x+z+u}{3}&=8 \qquad (3)\\ u+2x+2y+2z&=31 \qquad (4)\end{aligned}\right.$

Clearing (2) and (3) of fractions, $\begin{cases} 2x+y+z+u=20 & (5) \\ 3y+x+z+u=24 & (6) \end{cases}$

Subtract. (1) from (5), and from (6), $x=2$, and $y=3$

Subtract. (5) from (4), $y+z=11$, or $3+z=11$, and $z=8$

Subst. the values of $x$, $y$, and $z$ in (1), $2+3+8+u=18$, or $u=5$.

Prob. 37. Let $x$, $y$, $u$, and $z$ represent the four digits.

By the conditions,

$$\begin{cases} x+y+u+z=19 & (1) \\ x=y+u, \text{ or } x-y-u=0 & (2) \\ y=u+z, \text{ or } y-u-z=0 & (3) \\ 1000x+100y+10u+z-8082=1000z+100u+10y+x & (4) \end{cases}$$

Trans. terms of (4), and dividing by 9,

$$111x+10y-10u-111z=898 \quad (5)$$

Adding (1) and (3), $x+2y=19$ (6)

Mult. (3) by 10, and subtract. it from (5), $111x-101z=898$ (7)

Adding (1) and (2), $2x+z=19$ (8)

Mult. (8) by 101, and adding it to (7), $313x=2817$, or $x=9$

Subst. this value in (6) and (8), $9+2y=19$, or $y=5$

$18+z=19$, or $z=1$

Subst. the values of $x$, $y$, and $z$ in (1), $9+5+u+1=19$, or $u=4$

And the number sought is 9541.

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## RADICAL QUANTITIES.

Note.—The examples in Involution and many of those in Evolution are mere illustrations of the rules, and of so simple a nature that it is thought unnecessary to exhibit solutions of them.

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### Reduction of Radical Quantities.

Art. 283. Ex. 5. $\sqrt[4]{\frac{a^4b}{c^5d}}=\sqrt[4]{\frac{a^4}{c^4}}\times\sqrt[4]{\frac{b}{cd}}$. But $\sqrt[4]{\frac{a^4}{c^4}}=\frac{a}{c}$

Therefore, $\sqrt[4]{\frac{a^4b}{c^5d}}=\frac{a}{c}\sqrt[4]{\frac{b}{cd}}$.

Ex. 6. $\sqrt[n]{a^nb}=\sqrt[n]{a^n}\times\sqrt[n]{b}$. But $\sqrt[n]{a^n}=a$.
Therefore, $\sqrt[n]{a^nb}=a\sqrt[n]{b}$.

Ex. 7. $(a^3-a^2b)^{\frac{1}{2}}=\sqrt{a^2}\times(a-b)^{\frac{1}{2}}$. But $\sqrt{a^2}=a$.
Therefore, $(a^3-a^2b)^{\frac{1}{2}}=a(a-b)^{\frac{1}{2}}$.

Ex. 8. $(54a^6b)^{\frac{1}{3}}=\sqrt[3]{27a^6}\times(2b)^{\frac{1}{3}}$. But $\sqrt[3]{27a^6}=3a^2$.
Therefore, $(54a^6b)^{\frac{1}{3}}=3a^2(2b)^{\frac{1}{3}}$.

Ex. 9. $\sqrt{98a^2x}=\sqrt{49a^2}\times\sqrt{2x}$. But $\sqrt{49a^2}=7a$.
Therefore, $\sqrt{98a^2x}=7a\sqrt{2x}$.

Ex. 10. $\sqrt[3]{a^3+a^3b^2}-\sqrt[3]{a^3}\times\sqrt[3]{1+b^2}$. But $\sqrt[3]{a^3}=a$.
Therefore, $\sqrt[3]{a^3+a^3b^2}=a\sqrt[3]{1+b^2}$.

## Multiplication of Radical Quantities.

ART. 295. Ex. 1. Reducing the quantities to a common index, $\sqrt{a}=\sqrt[6]{a^3}$. $\sqrt[3]{b}=\sqrt[6]{b^2}$.

Therefore, $\sqrt{a}\times\sqrt[3]{b}=\sqrt[6]{a^3}\times\sqrt[6]{b^2}=\sqrt[6]{a^3b^2}$.

Ex. 2. $5\sqrt{5}\times3\sqrt{8}=15\sqrt{40}$. $\sqrt{40}=\sqrt{4}\times\sqrt{10}$.
But $\sqrt{4}=2$.
Then, $\sqrt{40}=2\sqrt{10}$. And $15\sqrt{40}=15\times2\sqrt{10}$, or $30\sqrt{10}$.

Ex. 3. $2\sqrt{3}=2\sqrt[6]{27}$. $3\sqrt[3]{4}=3\sqrt[6]{16}$.
Then, $2\sqrt{3}\times3\sqrt{4}=2\sqrt[6]{27}\times3\sqrt[6]{16}=6\sqrt[6]{432}$.

Ex. 4. $\sqrt{d}=\sqrt[6]{d^3}$. $\sqrt[3]{ab}=\sqrt[6]{a^2b^2}$.
Then, $\sqrt{d}\times\sqrt[3]{ab}=\sqrt[6]{d^3}\times\sqrt[6]{a^2b^2}=\sqrt[6]{a^2b^2d^3}$.

Ex. 5. $\sqrt{\frac{2ab}{3c}}\times\sqrt{\frac{9ad}{2b}}=\sqrt{\frac{18a^2bd}{6bc}}=\sqrt{\frac{3a^2d}{c}}$.

EXAMPLE 6.

$a\sqrt{a-x}$, mult. by $(c-d)\times\sqrt{ax}=(ac-ad)\times\sqrt{a^2x-ax^2}$.
For the rational parts, $a\times(c-d)$, give $(ac-ad)$;
And the radical parts, $\sqrt{a-x}\times\sqrt{ax}=\sqrt{a^2x-ax^2}$.

## Division of Radical Quantities.

ART. 299. Ex. 1. Reducing the quantities to a common index, $2\sqrt[3]{bc}=2\sqrt[6]{b^2c^2}$. $3\sqrt{ac}=3\sqrt[6]{a^3c^3}$.

Then, $2\sqrt[3]{bc}\div3\sqrt{ax}=2\sqrt[6]{b^2c^2}\div3\sqrt[6]{a^3c^3}=\frac{2}{3}\sqrt[6]{\frac{b^2c^2}{a^3c^3}}=\frac{2}{3}\sqrt[6]{\frac{b}{a^3c}}$.

Ex. 2. $10\sqrt[3]{108}\div5\sqrt[3]{4}=2\sqrt[3]{27}$. But $\sqrt[3]{27}=3$.
Therefore, $2\sqrt[3]{27}=2\times3=6$.

Ex. 3. $10\sqrt{27}\div2\sqrt{3}=5\sqrt{9}$. But $\sqrt{9}=3$.
Then, $5\sqrt{9}=5\times3=15$.

Ex. 4. $8\sqrt{108}\div2\sqrt{6}=4\sqrt{18}$.
But $\sqrt{18}=\sqrt{9}\times\sqrt{2}$. And $\sqrt{9}=3$.
Then, $\sqrt{18}=3\sqrt{2}$. And $4\sqrt{18}=4\times3\sqrt{2}=12\sqrt{2}$.

Ex. 5. $\sqrt{d}=\sqrt[6]{d^3}$.

Then, $(a^2b^2d^3)^{\frac{1}{6}}\div\sqrt[6]{d^3}=(a^2b^2)^{\frac{1}{6}}$, or $(ab)^{\frac{1}{3}}$.

Ex. 6. $2a=\sqrt{4a^2}$.

Then, $(16a^3-12a^2x)^{\frac{1}{2}}\div\sqrt{4a^2}=(4a-3x)^{\frac{1}{2}}$.

### Miscellaneous Examples.

ART. 310. Ex. 1. $\sqrt[4]{81a^2}=\sqrt{9a}$.
But $\sqrt{9a}=\sqrt{9}\times\sqrt{a}$. And $\sqrt{9}=3$.
Therefore, $\sqrt[4]{81a^2}\ \sqrt{9a}=3\sqrt{a}$.

Ex. 2. The 6th root of $(a+b)^{-3}=(a+b)^{-3\times\frac{1}{6}}=(a+b)^{-\frac{1}{2}}$.

Ex. 3. The $n$th root of $(x-y)^{\frac{1}{6}}=(x-y)^{\frac{1}{6}\times\frac{1}{n}}=(x-y)^{\frac{1}{6n}}$.

Ex. 4. $(-125a^3x^6)^{\frac{1}{3}}=-5ax^2$.

Ex. 5. The square root of $\frac{4a^4}{9x^2y^2}=\frac{2a^2}{3xy}$.

Ex. 6. The 5th root of $\frac{32a^5x^{10}}{243}=\frac{2ax^2}{3}$.

Ex. 7. By Art. 277, the sq. root of $x^2-6bx+9b^2=x-3b$.

Ex. 8. The square root of $a^2+ay+\frac{y^2}{4}=a+\frac{y}{2}$.

Ex. 9. $ax^2$, in the form of the 6th root, is $(a^6x^{12})^{\frac{1}{6}}$.

Ex. 10. $-3y$, in the form of the cube root, is $(-27y^3)^{\frac{1}{3}}$.

Ex. 11. $a^2$ and $a^{\frac{1}{3}}$, under a common index, are $(a^6)^{\frac{1}{3}}$ and $a^{\frac{1}{3}}$.

Ex. 12. $4^{\frac{1}{3}}$ and $5^{\frac{1}{4}}$, under a common index, are $(4^4)^{\frac{1}{12}}$ and $(5^3)^{\frac{1}{12}}$, or $(256)^{\frac{1}{12}}$ and $(125)^{\frac{1}{12}}$.

Ex. 13. $a^{\frac{1}{2}}$ and $b^{\frac{1}{4}}$, under the index $\frac{1}{8}$, are $(a^4)^{\frac{1}{8}}$ and $(b^2)^{\frac{1}{8}}$.

Ex. 14. $2^{\frac{1}{2}}$ and $4^{\frac{1}{4}}$, under the index $\frac{1}{8}$, are $(2^4)^{\frac{1}{8}}$ and $(4^2)^{\frac{1}{8}}$, or $(16)^{\frac{1}{8}}$ and $(16)^{\frac{1}{8}}$.

Ex. 15. $\sqrt{294}=\sqrt{49}\times\sqrt{6}$. But $\sqrt{49}=7$.
Then, $\sqrt{294}=7\sqrt{6}$.

Ex. 16. $\sqrt{x^3-a^2x^2}=\sqrt{x^2}\times\sqrt{x-a^2}$. But $\sqrt{x^2}=x$.
Then, $\sqrt{x^3-a^2x^2}=x\sqrt{x-a^2}$.

Ex. 17. $\sqrt{16a^2x}=\sqrt{16a^2}\times\sqrt{x}$. But $\sqrt{16a^2}=4a$.
Then, $\sqrt{16a^2x}=4a\sqrt{x}$. $\sqrt{4a^2x}=\sqrt{4a^2}\times\sqrt{x}$.
But $\sqrt{4a^2}=2a$. Then, $\sqrt{4a^2x}=2a\sqrt{x}$.
Then the difference of the quantities $=4a\sqrt{x}-2a\sqrt{x}=2a\sqrt{x}$.
And the sum of the quantities $=4a\sqrt{x}+2a\sqrt{x}=6a\sqrt{x}$.

Ex. 18. $\sqrt[3]{192}=\sqrt[3]{64}\times\sqrt[3]{3}$. But $\sqrt[3]{64}=4$.
Then, $\sqrt[3]{192}=4\sqrt[3]{3}$. $\sqrt[3]{24}=\sqrt[3]{8}\times\sqrt[3]{3}$.
But $\sqrt[3]{8}=2$. Then, $\sqrt[3]{24}=2\sqrt[3]{3}$.
Then the difference of the quantities $=4\sqrt[3]{3}-2\sqrt[3]{3}=2\sqrt[3]{3}$.
And the sum of the quantities $=4\sqrt[3]{3}+2\sqrt[3]{3}=6\sqrt[3]{3}$.

Ex. 19. $7\sqrt[3]{18}\times5\sqrt[3]{4}=35\sqrt[3]{72}$.

But $\sqrt[3]{72}=\sqrt[3]{8}\times\sqrt[3]{9}$. And $\sqrt[3]{8}=2$.

Then, $\sqrt[3]{72}=2\sqrt[3]{9}$. And $35\sqrt[3]{72}=35\times2\sqrt[3]{9}=70\sqrt[3]{9}$.

$$\begin{array}{lll} \text{Ex. 20.} & \text{Mult.} & 4+2\sqrt{2} \\ & \text{By} & \underline{2-\ \sqrt{2}} \\ & & 8+4\sqrt{2} \\ & & \underline{\quad-4\sqrt{2}-4\quad} \\ & \textit{Ans.} & 8 \qquad\quad -4=4 \end{array}$$

$$\begin{array}{lll} \text{Ex. 21.} & \text{Mult.} & a(a+\sqrt{c})^{\frac{1}{2}} \\ & \text{By} & \underline{b(a-\sqrt{c})^{\frac{1}{2}}} \\ & \textit{Ans.} & ab(a^2-c)^{\frac{1}{2}}. \end{array}$$

The product of the quantities in the parentheses is, by Art. 111, $(a^2-c)^{\frac{1}{2}}$.

### EXAMPLE 22.

$$2(a+b)^{\frac{1}{n}}\times3(a+b)^{\frac{1}{m}}=2(a+b)^{\frac{m}{nm}}\times3(a+b)^{\frac{n}{nm}}=6(a+b)^{\frac{m+n}{mn}}.$$

Ex. 23. $6\sqrt{54}\div3\sqrt{2}=2\sqrt{27}$.

But $\sqrt{27}=\sqrt{9}\times\sqrt{3}$. And $\sqrt{9}=3$.

Then, $\sqrt{27}=3\sqrt{3}$. And $2\sqrt{27}=2\times3\sqrt{3}=6\sqrt{3}$.

Ex. 24. $4\sqrt[3]{72}\div2\sqrt[3]{18}=2\sqrt[3]{4}$.

Ex. 25. $\sqrt{7}=\sqrt[6]{7^3}$. And $\sqrt[3]{7}=\sqrt[6]{7^2}$.

Then, $\sqrt[6]{7^3}\div\sqrt[6]{7^2}=\sqrt[6]{7}$.

Ex. 26. $8\sqrt[3]{512}\div4\sqrt[3]{2}=2\sqrt[3]{256}$.

$\sqrt[3]{256}=\sqrt[3]{64}\times\sqrt[3]{4}$. But $\sqrt[3]{64}=4$.

Then, $\sqrt[3]{256}=4\sqrt[3]{4}$. And $2\sqrt[3]{256}=2\times4\sqrt[3]{4}=8\sqrt[3]{4}$.

### EXAMPLE 27.

$$(17\sqrt{21})^3=17\sqrt{21}\times17\sqrt{21}\times17\sqrt{21}=103173\sqrt{21}.$$

### EXAMPLE 28.

The square of $5+\sqrt{2}$, by Art. 109, is $25+10\sqrt{2}+2$.

Ex. 29. Fourth power of $\frac{1}{6}\sqrt{6}$ is the fourth power of $\frac{1}{6}\times$ fourth power of $\sqrt{6}$. $(\frac{1}{6})^4=\frac{1}{1296}$. $(\sqrt{6})^4=36$. $\frac{1}{1296}\times 36=\frac{1}{36}$.

EXAMPLE 30.

The cube of $\sqrt{x}-\sqrt{b}=(\sqrt{x}-\sqrt{b})\times(\sqrt{x}-\sqrt{b})\times(\sqrt{x}-\sqrt{b})$.

$$(\sqrt{x}-\sqrt{b})\times(\sqrt{x}-\sqrt{b})=x-2\sqrt{bx}+b).$$

Then mult. by $\sqrt{x}-\sqrt{b}$ again,

$$(x-2\sqrt{bx}+b)\times(\sqrt{x}-\sqrt{b})=x\sqrt{x}-3x\sqrt{b}+3b\sqrt{x}-b\sqrt{b}.$$

Ex. 31. The factor which will make $y^{\frac{1}{3}}$ a rational quantity is $y^{\frac{2}{3}}$. For $y^{\frac{1}{3}}\times y^{\frac{2}{3}}=y^{\frac{1}{3}+\frac{2}{3}}=y^{\frac{3}{3}}$, or $y$.

Ex. 32. That factor is $\sqrt{5}+\sqrt{x}$. For, by Art. 111, we find that $(\sqrt{5}-\sqrt{x})\times(\sqrt{5}+\sqrt{x})=5-x$, which is a rational quantity.

Ex. 33. $\frac{\sqrt{a}}{\sqrt{x}}=\frac{\sqrt{a}\times\sqrt{a}}{\sqrt{x}\times\sqrt{a}}=\frac{a}{\sqrt{ax}}$. Here we multiply both numerator and denominator by $\sqrt{a}$, and this gives a rational numerator.

Ex. 34. $\frac{\sqrt{6}}{\sqrt{7}\times\sqrt{3}}=\frac{\sqrt{6}}{\sqrt{21}}=\frac{\sqrt{6}\times\sqrt{21}}{\sqrt{21}\times\sqrt{21}}=\frac{\sqrt{126}}{21}$. Here we multiply by the $\sqrt{21}$, which is the same as $\sqrt{7}\times\sqrt{3}$. This gives a rational denominator.

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## REDUCTION OF EQUATIONS BY INVOLUTION AND EVOLUTION.

Ex. 6. Reduce the equation $3+2\sqrt{x}-\frac{4}{5}=6$.

Transposing and uniting terms, $2\sqrt{x}=6+\frac{4}{5}-3=3\frac{4}{5}$, or $\frac{19}{5}$

Dividing by 2, and involving both sides, $x=(\frac{19}{10})^2=\frac{361}{100}$.

Ex. 7. Reduce $4\sqrt{\frac{x}{5}}=8$.

Dividing by 4, $\sqrt{\frac{x}{5}}=2$

Involving both sides, $\frac{x}{5}=4$, or $x=20$.

Ex. 8. Reduce $(2x+3)^{\frac{1}{3}}+4=7$.

Transposing, $(2x+3)^{\frac{1}{3}}=3$

Involving both sides, $2x+3=27$, or $2x=24$, and $x=12$.

Ex. 9. Reduce $\sqrt{12+x}=2+\sqrt{x}$.

Involving both sides, $12+x=4+4\sqrt{x}+x$

Transposing and uniting terms, $4\sqrt{x}=8$

Dividing by 4, $\sqrt{x}=2$, and $x=4$.

Ex. 10. Reduce $\sqrt{x-a}=\sqrt{x}-\frac{1}{2}\sqrt{a}$.

Involving both sides, $x-a=x-\sqrt{ax}+\frac{a}{4}$

Transposing and uniting terms, $\sqrt{ax}=\frac{5a}{4}$

Squaring each side, and div. by $a$, $ax=\frac{25a^2}{16}$, and $x=\frac{25a}{16}$.

Ex. 11. Reduce $\sqrt{5}\times\sqrt{x+2}=2+\sqrt{5x}$.

Involving both sides, $5x+10=4+4\sqrt{5x}+5x$

Transposing and uniting terms, $4\sqrt{5x}=6$, or $\sqrt{5x}=\frac{3}{2}$

Squaring each side, and dividing by 5, $5x=\frac{9}{4}$, and $x=\frac{9}{20}$.

Ex. 12. Reduce $\frac{x-ax}{\sqrt{x}}=\frac{\sqrt{x}}{x}$.

Multiplying by $\sqrt{x}$, $x-ax=\frac{\sqrt{x}\times\sqrt{x}}{x}=\frac{x}{x}=1$

Therefore, $x(1-a)=1$, and $x=\frac{1}{1-a}$.

Ex. 13. Reduce $\frac{\sqrt{x}+28}{\sqrt{x}+4}=\frac{\sqrt{x}+38}{\sqrt{x}+6}$.

Clearing of fractions, $x+34\sqrt{x}+168=x+42\sqrt{x}+152$

Trans. and uniting terms, $8\sqrt{x}=16$, or $\sqrt{x}=2$, and $x=4$.

Ex. 14. Reduce $\sqrt{x}+\sqrt{a+x}=\frac{2a}{\sqrt{a+x}}$.

Clearing of fractions, $\sqrt{ax+x^2}+a+x=2a$

Trans. and uniting terms, $\sqrt{ax+x^2}=a-x$

Squaring each side, $ax+x^2=a^2-2ax+x^2$

Trans. and uniting terms. $3ax=a^2$, or $x=\frac{a^2}{3a}=\frac{a}{3}$.

Ex. 15. Reduce $x+\sqrt{a^2+x^2}=\frac{2a^2}{\sqrt{a^2+x^2}}$.

Clear. of fractions, $x\sqrt{a^2+x^2}+a^2+x^2=2a^2$

Trans. and uniting terms, $x\sqrt{a^2+x^2}=a^2-x^2$

Involving both sides, $a^2x^2+x^4=a^4-2a^2x^2+x^4$

Trans. and uniting terms, $3a^2x^2=a^4$

Div. by $3a^2$, and extract. the root, $x^2=\frac{a^4}{3a^2}=\frac{1}{3}a^2$, and $x=a\sqrt{\frac{1}{3}}$.

Ex. 16. Reduce $x+a=\sqrt{a^2+x\sqrt{b^2+x^2}}$.

Squaring each side, $x^2+2ax+a^2=a^2+x\sqrt{b^2+x^2}$

Expunge $a^2$, and div. by $x$, $x+2a=\sqrt{b^2+x^2}$

Squaring each side, $x^2+4ax+4a^2=b^2+x^2$

Trans. and uniting terms, $4ax=b^2-4a^2$

Dividing by $4a$, $x=\frac{b^2-4a^2}{4a}$.

Ex. 17. Reduce $\sqrt{2+x}+\sqrt{x}=\frac{4}{\sqrt{2+x}}$.

Clearing of fractions, $2+x+\sqrt{2x+x^2}=4$

Trans. and uniting terms, $\sqrt{2x+x^2}=2-x$

Squaring each side, $2x+x^2=4-4x+x^2$

Trans. and uniting terms, $6x=4$, and $x=\frac{2}{3}$.

Ex. 18. Reduce $\sqrt{x-32}=16-\sqrt{x}$.

Squaring each side, $x-32=256-32\sqrt{x}+x$

Transposing and uniting terms, $32\sqrt{x}=288$.

Div. by 32, and squaring each side, $\sqrt{x}=9$, and $x=81$.

Ex. 19. Reduce $\sqrt{4x+17}=2\sqrt{x}+1$.

Involving both sides, $4x+17=4x+4\sqrt{x}+1$

Transposing and uniting terms, $4\sqrt{x}=16$

Div. by 4, and squaring each side, $\sqrt{x}=4$, and $x=16$.

Ex. 20. Reduce $\dfrac{\sqrt{6x}-2}{\sqrt{6x}+2}=\dfrac{4\sqrt{6x}-9}{4\sqrt{6x}+6}$.

Clearing of fractions,

$$24x-8\sqrt{6x}+6\sqrt{6x}-12=24x-9\sqrt{6x}+8\sqrt{6x}-18$$

Trans. and uniting terms, $\sqrt{6x}=6$

Squar. each side, and div. by 6, $6x=36$, and $x=6$.

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### Problems under Article 316.

Prob. 6. Let $x=$ the required number.

By the conditions, $8-\dfrac{x^2}{4}=4$

Transposing and changing signs, $\dfrac{x^2}{4}=4$

Extracting the root, $\dfrac{x}{2}=2$, and $x=4$.

Prob. 7. Let $10x=$ the sum of the two numbers: then $7x$ $=$ the greater, and $10x-7x=3x=$ the smaller number.

By the supposition, $10x\times 3x=30x^2=270$

Div. by 30, and extract. the root, $x^2=9$, and $x=3$

Therefore, $7x=21=$ the greater, and $3x=9=$ the less.

PROB. 8. Let $2x=$ the difference of the two numbers: then $9x=$ the greater, and $9x-2x=7x=$ the less.
By the supposition, $(9x)^2-(7x)^2=128$
Unit. terms, and div. by 32, $32x^2=128$, or $x^2=4$, and $x=2$
Therefore, $9x=18=$ the greater number, $7x=14=$ the less.

PROB. 10. Let $x=$ the greater part: then $14-x=$ the smaller.

By the conditions, $\frac{x}{14-x}:\frac{14-x}{x}::16:9$

Mult. extremes and means, $\frac{9x}{14-x}=\frac{16(14-x)}{x}$

Clearing of fractions, $9x^2=16(14-x)^2$
Extracting the square root of each side, $3x=4(14-x)$
Trans. and uniting terms, $7x=56$, and $x=8$, the greater,
And $14-8=6$, the less.

PROB. 11. Let $5x$ and $4x$ be the two numbers.
By the supposition, $125x^3+64x^3=189x^3=5103$
Div. and extract. the square root, $x^3=27$, and $x=3$
Therefore, $5x=15$, the greater, and $4x=12$, the smaller number.

PROB. 12. Let $x=$ the number of miles A travelled: then $x-18=$ the number of miles B travelled. If A could have gone B's distance $(x-18)$ in $15\frac{3}{4}$ days, then $\frac{x-18}{15\frac{3}{4}}$ is the distance he travels in one day. And if B could have gone A's distance $(x)$ in 28 days, then $\frac{x}{28}$ is the distance he travels in one day. But the whole distance each travels is proportioned to his daily progress;

Therefore, $x:x-18::\frac{x-18}{15\frac{3}{4}}:\frac{x}{28}$

Mult. extremes and means, $\frac{x^2}{28}=\frac{(x-18)^2}{15\frac{3}{4}}$

Clear. of fract. and mult. by 4, $63x^2=112(x-18)^2$
Dividing each side by 7, $9x^2=16(x-18)^2$
Extracting the square root, $3x=4(x-18)$

Or, $3x=4x-72$, and $x=72=$ A's distance.
Therefore, $72-18=54=$ B's distance,
$72+54=126=$ the whole distance.

PROB. 13. Let $8x$ and $5x=$ the two numbers.
By the conditions, $8x\times5x=40x^2=360$
Div. by 40, and extract. the root, $x^2=9$, and $x=3$
Hence $8\times3=24=$ the greater, and $5\times3=15=$ the less.

PROB. 14. Let $x=$ the number of yards in one piece, $36-x$ $=$ the number of yards in the other piece: then $(36-x)^2=$ the cost of one piece, and $x^2=$ the cost of the other.

By the conditions, $(36-x)^2 : x^2 :: 4 : 1$
Mult. extremes and means, $(36-x)^2=4x^2$
Extract. the square root, $36-x=2x$, or $3x=36$
Therefore, $x=12$, the length of one,
And $36-12=24$, the length of the other.

PROB. 15. Let $3x$ and $2x=$ the numbers.
By the supposition, $81x^4+16x^4 : 27x^3+8x^3 :: 26 : 7$
Unit. terms, and mult. extremes and means, $455x^4=910x^3$
Dividing by $455x^3$, $x=3$, and 6 and 4 are the numbers.

PROB. 16. Let $x=$ the number of gentlemen: then $x^2=$ the number of servants, and $x\times2x^2=2x^3=$ the whole amount of money.

By the supposition, $2x^3=3456$
Div. by 2, and extract. the root, $x^3=1728$, and $x=12$.

PROB. 17. Let $x=$ the number of companies: then $x\times4x$ $=4x^2=$ the number of men first drafted; and $x\times3=3x=$ the number drafted afterwards.

By the conditions, $4x^2 : 4x^2+3x :: 16 : 17$
Multiplying extremes and means, $68x^2=64x^2+48x$
Trans. and dividing by $4x$, $4x^2=48x$, or $x=12$.

## QUADRATIC EQUATIONS.

Every affected quadratic equation may be reduced to one of the four following forms, differing only in the signs.

$$\begin{array}{lr} x^2+ax=\phantom{-}b & (1)\\ x^2-ax=\phantom{-}b & (2)\\ x^2+ax=-b & (3)\\ x^2-ax=-b & (4)\end{array}$$

$$\text{The above when reduced become}\left\{\begin{array}{lr} x=-\frac{1}{2}a\pm\sqrt{\phantom{-}b+\frac{1}{4}a^2} & (1)\\ x=\phantom{-}\frac{1}{2}a\pm\sqrt{\phantom{-}b+\frac{1}{4}a^2} & (2)\\ x=-\frac{1}{2}a\pm\sqrt{-b+\frac{1}{4}a^2} & (3)\\ x=\phantom{-}\frac{1}{2}a\pm\sqrt{-b+\frac{1}{4}a^2} & (4)\end{array}\right.$$

In each of the above quadratic equations it will be observed that the value of the unknown quantity is equal to half the coefficient of the first power of $x$ with its sign changed, plus or minus the square root of the known term increased by the square of half the coefficient of the first power of $x$. Hence, if an equation of either of the above forms is given, the reduction may be regarded as already effected, and then the value of $x$ may be found at once by the following Rule:

*To and from half the coefficient of the first power of the unknown quantity, add and subtract the square root of the known term increased by the square of half the coefficient of the first power of the unknown quantity, and the results will be the two values of $x$.*

In many of the following solutions the processes of completing the square, extracting the root, &c., are not written out, but after the equation has been reduced to one of the above forms, the value of $x$ is found directly by the Rule just given.

Ex. 1. Reduce $3x^2-9x-4=80$.

Transposing, uniting, and dividing by 3, $x^2-3x=28$

Completing the square (Art. 329), $4x^2-12x+9=112+9$

Extracting the square root, $2x-3=\pm\sqrt{121}=\pm 11$

Trans. and unit. terms, $2x=3\pm 11$, $x=\frac{14}{2}=7$, or $x=-\frac{8}{2}=-4$.

Ex. 2. Reduce $4x-\frac{36-x}{x}=46.$

Clearing of fractions, $4x^2-36+x=46x$

Trans. and uniting terms, $4x^2-45x=36$

Complet. the sq. (Art. 329), $64x^2-720x+2025=576+2025$

Extracting the square root, $8x-45=\pm51$

Trans. unit. and div. $8x=45\pm51$, $x=\frac{96}{8}=12$, or $x=\frac{-6}{8}=-\frac{3}{4}$.

Ex. 3. Reduce $4x-\frac{14-x}{x+1}=14.$

Clearing of fractions, $4x^2+4x-14+x=14x+14$

Trans. and uniting terms, $4x^2-9x=28$

Completing the sq. (Art. 329), $64x^2-144x+81=448+81$

Extracting the square root, $8x-9=\pm23$

Trans. unit. and div. $8x=9\pm23$, $x=\frac{32}{8}=4$, or $x=-\frac{14}{8}=-\frac{7}{4}$.

Ex. 4. Reduce $5x-\frac{3x-3}{x-3}=2x+\frac{3x-6}{2}.$

Clearing of fractions,

$$10x^2-30x-6x+6=4x^2-12x+3x^2-9x-6x+18$$

Transposing and uniting terms, $3x^2-9x=12$

Div. by 3, and comp. the sq. (Art. 322), $x^2-3x+\frac{9}{4}=4+\frac{9}{4}=\frac{25}{4}$

Extracting the square root. $x-\frac{3}{2}=\pm\sqrt{\frac{25}{4}}=\pm\frac{5}{2}$

Trans. and unit. $x=\frac{3}{2}\pm\frac{5}{2}$, $x=\frac{8}{2}=4$, or $x=-\frac{2}{2}=-1$.

Ex. 5. Reduce $\frac{16}{x}-\frac{100-9x}{4x^2}=3.$

Mult. by $4x^2$, $64x-100+9x=12x^2$

Trans. and unit. terms, $-100=12x^2-73x$

Completing the sq. $576x^2-3504x+5329=-4800+5329$

Extracting the sq. root, $24x-73=\pm23$

Trans. unit. and div. $24x=73\pm23$, $x=\frac{96}{24}=4$, or $x=\frac{50}{24}=2\frac{1}{12}$.

Ex. 6. Reduce $\frac{3x-4}{x-4}+1=10-\frac{x-2}{2}.$

Clear. of fract. $6x-8+2x-8=20x-80-x^2+2x+4x-8$

Transposing and uniting terms, $x^2-18x=-72$

Find. the value of $x$ (see Rule given above), $x=9\pm\sqrt{-72+81}$

Or, $x=9\pm\sqrt{9}=9\pm3=12$ or $6$

Ex. 7. Reduce $\frac{x+4}{3}-\frac{7-x}{x-3}=\frac{4x+7}{9}-1$.

Clearing of fractions,

$3x^2-9x+12x-36-63+9x=4x^2+7x-12x-21-9x+27$

Trans. and uniting terms, $x^2-26x=-105$

Finding the value of $x$ (see Rule given above),

$x=13\pm\sqrt{-105+169}$

Or, $x=13\pm\sqrt{64}=13\pm8=21$ or $5$.

Ex. 8. Reduce $\frac{x^3-10x^2+1}{x^2-6x+9}=x-3$.

Clear. of fract. $x^3-10x^2+1=x^3-3x^2-6x^2+18x+9x-27$

Trans. and uniting terms, $x^2+27x=28$

Completing the sq. (Art. 329), $4x^2+108x+729=112+729$

Extracting the square root, $2x+27=\pm29$

Trans., uniting, and dividing, $2x=-27\pm29$, $x=\frac{2}{2}=1$

Or, $x=-\frac{56}{2}=-28$.

Ex. 9. Reduce $\frac{6}{x+1}+\frac{2}{x}=3$.

Clearing of fractions, $6x+2x+2=3x^2+3x$

Trans. and uniting terms, $3x^2-5x=2$

Completing the square, $36x^2-60x+25=24+25=49$

Extracting the root, $6x-5=\pm7$

And $6x=5\pm7=12$ or $-2$, and $x=\frac{12}{6}=2$, or $x=-\frac{2}{6}=-\frac{1}{3}$.

Ex. 10. Reduce $\frac{3x}{x+2}-\frac{x-1}{6}=x-9$.

Clear. of fract. $18x-x^2+x-2x+2=6x^2+12x-54x-108$

Trans. and unit. terms, $7x^2-59x=110$

Completing the square, $196x^2-1652x+3481=3080+3481$

Extracting the sq. root, $14x-59=\pm\sqrt{6561}=\pm81$

And $14x=59\pm81$, $x=\frac{140}{14}=10$, or $x=-\frac{22}{14}=-\frac{11}{7}$.

Ex. 11. Reduce $\frac{x}{a}+\frac{a}{x}=\frac{2}{a}$.

Multiplying by $ax$, $x^2+a^2=2x$, or $x^2-2x=-a^2$

Find. the value of $x$ by the Rule given above, $x=1\pm\sqrt{1-a^2}$.

Ex. 12. Reduce $x^4+ax^2=b$.

Completing the square (Art. 332), $x^4+ax^2+\frac{a^2}{4}=b+\frac{a^2}{4}$

Extracting the square root,

$$x^2+\frac{a}{2}=\pm\sqrt{b+\frac{a^2}{4}}, \text{ and } x^2=-\frac{a}{2}\pm\sqrt{b+\frac{a^2}{4}}$$

Extracting the root, $x=\left(-\frac{a}{2}\pm\sqrt{b+\frac{a^2}{4}}\right)^{\frac{1}{2}}$.

Ex. 13. Reduce $\frac{x^6}{2}-\frac{x^3}{4}=-\frac{1}{32}$.

Making $x^3=y$, and mult. by 2, $y^2-\frac{y}{2}=-\frac{1}{16}$

By the Rule given above, $y=\frac{1}{4}\pm\sqrt{-\frac{1}{16}+\frac{1}{16}}=\frac{1}{4}=x^3$

Restoring the value of $y$, and extract. the cube root, $x=\sqrt[3]{\frac{1}{4}}$.

Ex. 14. Reduce $2x^{\frac{2}{3}}+3x^{\frac{1}{3}}=2$.

Completing the square, $16x^{\frac{2}{3}}+24x^{\frac{1}{3}}+9=16+9=25$

Extracting the square root, $4x^{\frac{1}{3}}+3=\pm5$, or $4x^{\frac{1}{3}}=2$ or $-8$

Div. by 4, and cubing each side, $x^{\frac{1}{3}}=\frac{1}{2}$ or $-2$, and $x=\frac{1}{8}$ or $-8$.

Ex. 15. Reduce $\frac{1}{2}x-\frac{1}{3}\sqrt{x}=22\frac{1}{6}$.

Mult. by 2, and comp. the square, $x-\frac{2}{3}\sqrt{x}+\frac{1}{9}=44\frac{1}{3}+\frac{1}{9}$

Extract. the square root, $\sqrt{x}-\frac{1}{3}=\pm\sqrt{\frac{133}{3}+\frac{1}{9}}=\pm\sqrt{\frac{399+1}{9}}$

And $\sqrt{x}=\frac{1}{3}\pm\sqrt{\frac{400}{9}}=\frac{1}{3}\pm\frac{20}{3}=7$ or $-\frac{19}{3}$, and $x=49$ or $\frac{361}{9}$.

Ex. 16. Reduce $2x^4-x^2+96=99$.

Trans. and dividing by 2, $x^4-\frac{x^2}{2}=\frac{3}{2}$

Completing the square, $x^4-\frac{1}{2}x^2+\frac{1}{16}=\frac{3}{2}+\frac{1}{16}$

Extract. the square root, $x^2-\frac{1}{4}=\pm\sqrt{\frac{3}{2}+\frac{1}{16}}=\pm\sqrt{\frac{24+1}{16}}$

Therefore, $x^2=\frac{1}{4}\pm\frac{5}{4}=\frac{6}{4}$ or $-1$, and $x=\sqrt{\frac{6}{4}}=\frac{1}{2}\sqrt{6}$ or $\sqrt{-1}$.

Ex. 17. Reduce $(10+x)^{\frac{1}{2}}-(10+x)^{\frac{1}{4}}=2$.

Making $(10+x)^{\frac{1}{4}}$ equal $y$, $y^2-y=2$

By the Rule given above, $y=\frac{1}{2}\pm\sqrt{2+\frac{1}{4}}=\frac{1}{2}\pm\sqrt{\frac{9}{4}}$

Or, $y=\frac{1}{2}\pm\frac{3}{2}=2$ or $-1$, and $(10+x)^{\frac{1}{4}}=2$ or $-1$

Involving to the 4th power, $10+x=16$, and $x=6$

Also, $10+x=1$, and $x=-9$.

Ex. 18. Reduce $3x^{2n}-2x^n=8$.

Completing the square, $36x^{2n}-24x^n+4=96+4$

Extracting the square root, $6x^n-2=\pm10$

Trans. div. and tak. the $n$th root, $6x^n=12$ or $-8$, $x^n=2$ or $-\frac{4}{3}$

And $x=\sqrt[n]{2}$ or $\sqrt[n]{-\frac{4}{3}}$.

Ex. 19. Reduce $2(1+x-x^2)-\sqrt{1+x-x^2}=-\frac{1}{9}$.

Mak. $(1+x-x^2)^{\frac{1}{2}}=y$, and comp. the sq. $16y^2-8y+1=-\frac{8}{9}+1$

Extracting the root, $4y-1=\pm\frac{1}{3}$, and $y=\dfrac{3\pm1}{4\times3}=\frac{1}{3}$ or $\frac{1}{6}$

Subst. the 1st value of $y$, $y=(1+x-x^2)^{\frac{1}{2}}=\frac{1}{3}$, or $1+x-x^2=\frac{1}{9}$

Trans. and uniting terms, $x^2-x=\frac{8}{9}$

By the Rule given above, $x=\frac{1}{2}\pm\sqrt{\frac{8}{9}+\frac{1}{4}}=\frac{1}{2}\pm\sqrt{\frac{41}{36}}=\frac{1}{2}\pm\frac{1}{6}\sqrt{41}$

Subst. the 2d value of $y$, $y=(1+x-x^2)^{\frac{1}{2}}=\frac{1}{6}$, or $1+x-x^2=\frac{1}{36}$

Trans. and uniting terms, $x^2-x=\frac{35}{36}$

By the Rule given above,

$$x=\frac{1}{2}\pm\sqrt{\frac{35}{36}+\frac{1}{4}}=\frac{1}{2}\pm\sqrt{\frac{35+9}{36}}=\frac{1}{2}\pm\frac{1}{6}\sqrt{44}.$$

Ex. 20. Reduce $\sqrt[3]{x^3-a^3}=x-b$.

Cubing each side, $x^3-a^3=x^3-3x^2b+3xb^2-b^3$

Trans. and uniting terms, $3bx^2-3b^2x=a^3-b^3$

Dividing by 36, $x^2-bx=\dfrac{a^3-b^3}{3b}$

By the Rule given above,

$$x=\frac{b}{2}\pm\sqrt{\frac{a^3-b^3}{3b}+\frac{b^2}{4}}=\frac{b}{2}\pm\sqrt{\frac{4a^3-b^3}{12b}}.$$

Ex. 21. Reduce $\frac{\sqrt{4x}+2}{4+\sqrt{x}}=\frac{4-\sqrt{x}}{\sqrt{x}}$.

Clearing of fract. $2x+2\sqrt{x}=16-x$, or $3x+2\sqrt{x}=16$
Comp. the square, $36x+24\sqrt{x}+4=192+4=196$
Extracting the root, $6\sqrt{x}+2=\pm 14$, $6\sqrt{x}=-2\pm 14$
And $\sqrt{x}=2$ or $-\frac{8}{3}$, and $x=4$ or $\frac{64}{9}$.

Ex. 22. Reduce $x^{\frac{6}{5}}+x^{\frac{3}{5}}=756$.

Completing the square, $x^{\frac{6}{5}}+x^{\frac{3}{5}}+\frac{1}{4}=756+\frac{1}{4}=\frac{3025}{4}$
Extracting the square root, $x^{\frac{3}{5}}+\frac{1}{2}=\pm\frac{55}{2}$
Transposing and uniting, $x^{\frac{3}{5}}=-\frac{1}{2}\pm\frac{55}{2}=27$ or $-28$
By evolution and involution, $x^{\frac{1}{5}}=3$, and $x=243$.

Ex. 23. Reduce $\sqrt{2x+1}+2\sqrt{x}=\frac{21}{\sqrt{2x+1}}$.

Clear. of fract. $2x+1+2\sqrt{2x^2+x}=21$
Trans. and div. by 2, $\sqrt{2x^2+x}=10-x$
Squaring each side, $2x^2+x=100-20x+x^2$
Trans. and uniting, $x^2+21x=100$
By the Rule given above,

$$x=-\tfrac{21}{2}\pm\sqrt{100+(\tfrac{21}{2})^2}=-\tfrac{21}{2}\pm\sqrt{\frac{400+441}{4}}$$

$$=-\tfrac{21}{2}\pm\sqrt{\tfrac{841}{4}}=-\frac{21\pm 29}{2}=4 \text{ or } -25.$$

Ex. 24. Reduce $2\sqrt{x-a}+3\sqrt{2x}=\frac{7a+5x}{\sqrt{x-a}}$.

Clear. of fract. $2x-2a+3\sqrt{2x^2-2ax}=7a+5x$
Trans. and uniting, $3\sqrt{2x^2-2ax}=9a+3x$
Div. by 3, and squar. each side, $2x^2-2ax=9a^2+6ax+x^2$
Trans. and uniting, $x^2-8ax=9a^2$
By the Rule given above,

$$x=4a\pm\sqrt{9a^2+16a^2}=4a\pm\sqrt{25a^2}=4a+5a=9a.$$

Ex. 25. Reduce $x+16-7\sqrt{x+16}=10-4\sqrt{x+16}$.

Mak. $y=\sqrt{x+16}$, & subst. $y^2-7y=10-4y$, or $y^2-3y=10$

By the Rule given above,

$$y=\tfrac{3}{2}\pm\sqrt{10+\tfrac{9}{4}}=\tfrac{3}{2}\pm\sqrt{\tfrac{49}{4}}=\tfrac{3}{2}\pm\tfrac{7}{2}=5 \text{ or } -2$$

And since $y=\sqrt{x+16}=5$, $x+16=25$, and $x=9$.

Ex. 26. Reduce $\sqrt{x^5}+\sqrt{x^3}=6\sqrt{x}$.

Dividing by $\sqrt{x}$. $x^2+x=6$

Completing the square, $x^2+x+\frac{1}{4}=6+\frac{1}{4}=\frac{25}{4}$

Extracting the square root, $x+\frac{1}{2}=\pm\frac{5}{2}$, and $x=\dfrac{-1+5}{2}=2$.

Ex. 27. Reduce $\dfrac{4x-5}{x}-\dfrac{3x-7}{3x+7}=\dfrac{9x+23}{13x}$.

Mult. by $13x$, $52x-65-\dfrac{39x^2-91x}{3x+7}=9x+23$

Trans. and uniting, $43x-\dfrac{39x^2-91x}{3x+7}=88$

Mult. by $3x+7$, $129x^2+301x-39x^2+91x=264x+616$

Trans. and uniting, $90x^2+128x=616$

Dividing by 2, $45x^2+64x=308$

Comp. the square, $8100x^2+11520x+4096=55440+4096$

Extracting the root, $90x+64=\sqrt{59536}$,

$90x=-64+244$, and $x=2$.

Ex. 28. Reduce $\dfrac{3}{6x-x^2}+\dfrac{6}{x^2+2x}=\dfrac{11}{5x}$.

Mult. by $x$, $\dfrac{3}{6-x}+\dfrac{6}{x+2}=\dfrac{11}{5}$

Clear. of fract. $15x+30+180-30x=66x+132-11x^2-22x$

Trans. and unit. terms, $11x^2-59x=-78$

Comp. the sq. $484x^2-2596x+3481=-3432+3481$

Extract. the root, $22x-59=7$, or $22x=66$, and $x=3$.

Ex. 29. Reduce $(x-5)^3-3(x-5)^{\frac{3}{2}}=40$.

Comp. the square, $4(x-5)^3-12(x-5)^{\frac{3}{2}}+9=160+9=169$

Extracting the root, $2(x-5)^{\frac{3}{2}}-3=13$

Trans. and dividing by 2, $(x-5)^{\frac{3}{2}}=8$

Extracting the cube root, $(x-5)^{\frac{1}{2}}=2$

Squaring each side, $x-5=4$, and $x=9$.

Ex. 30. Reduce $x+\sqrt{x+6}=2+3\sqrt{x+6}$.

Transposing and uniting, $x-2=2\sqrt{x+6}$

Squaring each side, $x^2-4x+4=4x^2+24$

Transposing and uniting, $x^2-8x=20$

Completing the square, $x^2-8x+16=20+16=36$

Extracting the root, $x-4=6$, and $x=10$

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### Problems producing Quadratic Equations.

Prob. 2. Let $x$ and $45-x=$ the ages of the two brothers.

By the supposition, $(45-x)x=500$, or $x^2-45x=-500$

Completing the square, $4x^2-180x+2025=-2000+2025$

Extract. the sq. root, $2x-45=\sqrt{25}=5$, $2x=50$, $x=25$.

Therefore, $25=$ the age of one,

And $45-25=20$ the age of the other.

Prob. 6. Let $x$ and $x+12=$ the two numbers.

By the supposition, $x^2+(x+12)^2=1424$

Transposing and uniting terms, $2x^2+24x=1424-144=1280$

Div. by 2, and comp. the sq. $x^2+12x+36=640+36=676$

Extracting the square root, $x+6=26$, and $x=20$.

Therefore the two numbers are 20 and 32.

Prob. 7. Let $x$, and $x+120$, equal the two prizes.

By the conditions, $x+120 : x :: x : 10$

Mult. extremes and means, $10x+1200=x^2$, or $x^2-10x=1200$

By the Rule given above, $x=5\pm\sqrt{1200+25}=5\pm35=40$.

Therefore the prizes are 40 and 160.

**Prob.** 8. Let $x$ and $6-x$ be the two numbers.
By the cond. $x^3+(6-x)^3=x^3+216-108x+18x^2-x^3=72$
Transposing and uniting, $18x^2-108x=-144$
Dividing by 18, $x^2-6x=-8$
Completing the square, $x^2-6x+9=-8+9=1$
Extract. the sq. root, $x-3=1$, and $x=4$, or one number,
And $6-4=2$, the other number.

**Prob.** 9. Let $x=$the smaller part, and $56-x=$the greater.
By the conditions, $x(56-x)=640$, or $x^2-56x=-640$
Comp. the square, $x^2-56x+784=-640+784=144$
Extract. the square root, $x^2-28=\pm 12$, and $x=40$ or 16.
Therefore, $16=$the smaller, and $56-16=40=$the greater part.

**Prob.** 10. Let $x=$ the number of pieces: then $\dfrac{675}{x}=$ the cost of one piece, and $48x=$ the amount for which he sold the cloth.

By the conditions, $48x-675=\dfrac{675}{x}$
Clear. of fract. and div. by 3, $16x^2-225x=225$
Comp. the square, $1024x^2-14400x+50625=14400+50625$
Extracting the root, $32x-225=255$, $32x=480$, and $x=15$.

**Prob.** 11. Let $x=$ B's hourly progress, and $x+3=$ A's hourly progress: then $\dfrac{150}{x}=$ the number of hours B travels, and $\dfrac{150}{x+3}=$ the number of hours A travels.

By the supposition, $\dfrac{150}{x+3}+8\tfrac{1}{3}=\dfrac{150}{x}$
Clearing of fractions, $450x+25x^2+75x=450x+1350$
Trans. unit. and div. by 25, $x^2+3x=54$
Completing the square, $4x^2+12x+9=216+9=225$
Extracting the square root, $2x+3=15$, $2x=12$, and $x=6$.

Therefore B's hourly progress is 6 miles, and A's 9 miles.

Prob. 12. Let $x$ and $x+6$ be the two numbers.

By the conditions, $2x^2+47=(x+6)^2=x^2+12x+36$

Trans. and uniting terms, $x^2-12x=-11$

Completing the square, $x^2-12x+36=-11+36=25$

Extracting the square root, $x-6=5$, and $x=11$.

And $11+6=17=$ the greater number.

Prob. 13. Let $x=$ the number of persons B relieved, and $x+40=$ the number of persons A relieved: then $\frac{1200}{x+40}=$ what A gave to each, and $\frac{1200}{x}=$ what B gave to each.

By the supposition, $\frac{1200}{x+40}+5=\frac{1200}{x}$

Clearing of fractions, $1200x+5x^2+200x=1200x+48000$

Trans. unit. and div. by 5, $x^2+40x=9600$

By Rule giv. above, $x=-20\pm\sqrt{9600+400}=-20+100=80$

Therefore 80 are relieved by B, and 120 by A.

Prob. 14. Let $x$ and $10-x$ be the two numbers.

By the conditions, $x^2+100-20x+x^2=58$

Trans. unit. and div. by 2, $x^2-10x=-21$

By the Rule given above, $x=5\pm\sqrt{-21+25}=5\pm2=7$ or 3.

And the numbers are 7 and 3.

Prob. 15. Let $x=$ the number of gentlemen: then $\frac{175}{x}=$ each one's share if all had paid, and $\frac{175}{x-2}=$ what each one actually paid.

By the supposition, $\frac{175}{x}+10=\frac{175}{x-2}$

Clearing of fractions, $175x-350+10x^2-20x=175x$

Trans. uniting, and dividing by 10, $x^2-2x=35$

By the Rule given above, $x=2\pm\sqrt{35+1}=1\pm6=7.$

**Prob.** 16. Let $x=$ the number of yards purchased: then $\frac{60}{x}=$ the price paid per yard, and $\frac{54}{x-15}=$ the price received per yard: also 10 cents, or $\frac{1}{10}$ of a dollar, is the gain per yard.

By the supposition, $\frac{60}{x}+\frac{1}{10}=\frac{54}{x-15}$
Clearing of fractions, $600x-9000+x^2-15x=540x$
Transposing and uniting, $x^2-45x=9000$
Completing the square, $4x^2+180x+2025=36000+2025$
Extracting the root, $2x+45=\sqrt{38025}=195$
$2x=150$, and $x=75$.

Then $\frac{60}{75}=\frac{4}{5}$ of a dollar $=$ 80 cents, the price per yard.

**Prob.** 17. Let $x=$ the number of days which both travelled before they met: then $x-3=$ B's daily progress; also $(x-3)\times x=$distance B travelled, and $9x=$distance A travelled.

By the conditions, $(x-3)x+9x=247$, or $x^2+6x=247$
By Rule given above, $x=-3\pm\sqrt{247+9}=-3+16=13$ days
Therefore, $13\times 9=117=$the distance A travels
And $247-117=130=$ " " B "

**Prob.** 18. Let $x=$ the number of yards in the finer, and $x+2=$ the number of yards in the coarser piece: then $\frac{18}{x}$ pounds or $\frac{360}{x}$ shillings is the price per yard of the finer, and $\frac{16}{x+2}$ pounds or $\frac{320}{x+2}$ shillings is the price per yard of the coarser piece.

By the conditions, $\frac{320}{x+2}=\frac{360}{x}-4$
Clearing of fractions, $320x=360x+720-4x^2-8x$
Trans. unit. and dividing by 4, $x^2-8x=180$
By the Rule given above, $x=4\pm\sqrt{180+16}=4\pm14=18.$

Therefore $18=$ the number of yards in the finer piece, and $18+2=20=$ the number of yards in the coarser: and the prices were respectively 20 and 16 shillings per yard.

Prob. 19. Let $2x=$ the number of gallons of Teneriffe: then $x=$ the cost of the Madeira per gallon, and $x-4=$ the cost of the Teneriffe per gallon. $54+2x=$ the number of gallons in the mixture. Also, $54x=$ the whole cost of the Madeira, and $2x(x-4)=$ the whole cost of the Teneriffe.

By the conditions, $54x+2x(x-4)=10(54+2x)+576$*

Or, $54x+2x^2-8x=540+20x+576$

Tran. unit. and div. by 2, $x^2+13x=558$

Completing the square, $4x^2+52x+169=2232+169=2401$

Extracting the square root, $2x+13=49$, and $2x=36$.

Therefore $36=$ the number of gallons of the Teneriffe, and $x=18=$ the cost of the Madeira per gallon.

Prob. 20. Let $x=$ the number required.

By the conditions, $4=\frac{2(\sqrt{40-x^2}+10)}{x}$

Mult. by $x$, and div. by 2, $2x=\sqrt{40-x^2}+10$

Trans. and sq. each side, $4x^2-40x+100=40-x^2$

Trans. and uniting, $5x^2-40x=-60$

Div. by 5, and comp. the sq. $x^2-8x+16=-12+16=4$

Extracting the square root, $x-4=2$, and $x=6$.

Prob. 21. Let $x=$ the person's age.

By the conditions, $\frac{x}{2}+\sqrt{x}-12=0$

Transposing, and multiplying by 2, $x+2\sqrt{x}=24$

Completing the square, $x+2\sqrt{x}+1=24+1=25$

Extract. the square root, $\sqrt{x}+1=5$, or $\sqrt{x}=4$, and $x=16$.

Prob. 22. Let $x=$ the number of gallons in the smaller cask: then $x+5=$ the number of gallons in the larger cask, and $2x+5=$ the number of gallons in both casks. $\frac{58}{2x+5}=$ the price per gallon; but $\frac{x}{3}-2$ also equals the price per gallon.

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* 576 shillings = £28 16s.

Therefore, $\frac{58}{2x+5}=\frac{x}{3}-2$

Clearing of fractions, $174=2x^2+5x-12x-30$

Transposing and uniting, $2x^2-7x=204$

Completing the square, $16x^2-56x+49=1632+49=1681$

Extract. the square root, $4x-7=41$, $4x=48$, and $x=12$.

Therefore, $12 =$ the number of gallons in the smaller cask, and $12+5=17 =$ the number of gallons in the larger cask. $58\div 29=2 =$ the price per gallon.

**Prob.** 23. Let $x =$ the number of copper coins: then $24-x$ $=$ the number of silver coins.

By the conditions, $x(24-x)=24x-x^2=$ value of the copper

And $(24-x)x=24x-x^2=$ " " silver

And $48x-2x^2=216$, or $2x^2-48x=-216$

Div. by 2, and comp. the sq. $x^2-24x+144=-108+144=36$

Extracting the square root, $x-12=6$, $x=18$.

Therefore, 18 is the number of copper coins, and $24-x=6$ $=$ the number of silver coins.

**Prob.** 24. Let $x =$ the number of oxen: then $\frac{80}{x} =$ the cost of each ox. If he had bought 4 more oxen for the same money, then would $\frac{80}{x+4} =$ the cost of each ox.

By the conditions, $\frac{80}{x}=\frac{80}{x+4}+1$

Clearing of fractions, $80x+320=80\div x^2+4x$

Transposing and uniting, $x^2+4x=320$

Completing the square, $x^2+4x+4=324$

Extracting the square root, $x+2=18$, and $x=16$.

**Prob.** 25. Let $x=$ the smaller number, and $y=$the greater.

By the conditions, $3y-x=35$ (1)

And $\frac{4y}{3x+1}=x$ (2)

In equation (1) trans. $x$, and div. by 3, $y=\frac{35+x}{3}$ (3)

Clearing (2) of fractions, $4y=3x^2+x$ (4)

Div. (4) by 4, and equating the values of $y$, $\frac{35+x}{3}=\frac{3x^2+x}{4}$

Clearing of fractions, $140+4x=9x^2+3x$

Transposing and uniting, $9x^2-x=140$

Comp. the square, $324x^2-36x+1=5040+1=5041$

Extract. the square root, $18x-1=71$, $18x=72$, and $x=4$

Substituting this value in (3), $y=\frac{35+4}{3}=\frac{39}{3}=13$.

PROB. 26. Let $x$ and $y$ be the two numbers.

By the conditions, $x-y : x+y :: 2 : 3$ (1)

And $x+y : xy :: 3 : 5$ (2)

Adding and subtracting terms of (1), $2x : 2y :: 5 : 1$ (3)

Mult. extremes and means of (3), $2x=10y$, or $x=5y$

Substituting this value in (2), $6y : 5y^2 :: 3 : 5$

Multiplying extremes and means, $30y=15y^2$

Dividing both sides by $15y$, $y=2$, and $x=10$.

PROB. 27. Given $\frac{1}{3}(4x+2y)=6$ (1)

And $5xy=50$ (2)

Multiplying (1) by 3, $4x+2y=18$ (3)

Dividing (3) by 4, and transposing $y$, $x=\frac{18-2y}{4}=\frac{9-y}{2}$ (4)

Dividing (2) by $5y$, $x=\frac{50}{5y}=\frac{10}{y}$

Equating these values of $x$, $\frac{9-y}{2}=\frac{10}{y}$

Clearing of fractions, $9y-y^2=20$, or $y^2-9y=-20$

Comp. the square, $4y^2-36y+81=-80+81=1$

Extract. the sq. root, $2y-9=\pm1$, $2y=10$ or 8, and $y=5$ or 4

Subst. these values in (4), $x=\frac{9-5}{2}$ or $\frac{9-4}{2}=2$ or $2\frac{1}{2}$.

**Prob.** 28. Given $x+y=a$ (1)

And $x^2+y^2=b$ (2)

Squaring equation (1), $x^2+2xy+y^2=a^2$ (3)

Subtracting (2) from (3), $2xy=a^2-b$ (4)

Subtracting (4) from (2), $x^2-2xy+y^2=2b-a^2$

Extracting the square root, $x-y=\pm\sqrt{2b-a^2}$

Therefore (Art. 218), $x=\frac{a\pm\sqrt{2b-a^2}}{2}$, and $y=\frac{a\mp\sqrt{2b-a^2}}{2}$.

The examples under the subject of Ratio are so simple, that it is deemed unnecessary to present solutions of them.

◆

## PROPORTION.

Solutions which are given in the Algebra are not repeated here; but in some cases problems solved in the book are here wrought by a different method.

Ex. 1. Let $x=$ the greater, and $y=$ the less.

By the conditions proposed, $x+y=49$

And $x+6 : y-11 :: 9 : 2$

Multiplying extremes and means, $2x+12=9y-99$

Transposing and uniting terms, $9y-2x=111$

Multiplying the first equation by 2, $2y+2x=98$

Adding the last two equations, $11y=209$

Therefore, $y=19$, and $x=30$.

Ex. 3. Let $x=$ half their sum, and $y=$ half their difference.

By the conditions, $x+y : x-y :: 2x : 42$ (1)

And $x+y : x-y :: 2y : 6$ (2)

By equality of ratios, $2x : 42 :: 2y : 6$

By alternating and dividing terms, $x : y :: 21 : 3$

Mult. extremes and means, $3x=21y$, and $x=7y$

Mult. extremes and means of (2), $6x+6y=2xy-2y^2$

Substituting the value of $x$, $42y+6y=14y^2-2y^2$

Uniting terms, and div. by $12y$, $y=4$, and $x=28$

Therefore, $x+y=32=$ the greater, and $x-y=24=$ the less.

Ex. 4. Let $2x=$ the difference of the parts: then $(9+x)$ $=$ the greater, and $(9-x)=$ the less.
By the conditions, $(9+x)^2:(9-x)^2::25:16$
Extracting the square root, $9+x:9-x::5:4$
Multiplying extremes and means, $36+4x=45-5x$
Transposing and uniting terms, $9x=9$, and $x=1$
Therefore, $9+x=10=$the greater, and $9-x=8=$the less.

Ex. 6. Let $x=$ the greater, and $y=$ the less.
By the conditions, $x+y=20$ (1)
And $x:y::9:1$
By adding terms, $x+y:y::10:1$
Multiplying extremes and means, $x+y=10y$ (2)
Equating (1) and (2), $10y=20$, $y=2$, and $x=18$
A mean proportional between 18 and 2 $=\sqrt{2\times18}=6$.

Ex. 7. Let $2x=$ the sum, $2y=$ the difference: then $x+y$ $=$ the greater, and $x-y=$ the less.
By the conditions, $x^2-y^2=24$
And $6x^2y+2y^3:8y^3::19:1$
Multiplying the antecedents by 4, $24x^2y+8y^3:8y^3::76:1$
Div. 1st couplet by $y$, and subtract. terms, $24x^2:8y^2::75:1$
Dividing terms, $x^2:y^2::25:1$
Extract. and mult. extremes and means, $x=5y$, and $y=1$
Therefore, $x+y=6=$the greater, and $x-y=4=$the less.

Ex. 10. Let $x=$ the greater, and $y=$ the less.
By the conditions, $xy=135$ (1)
And $x^2-y^2:(x-y)^2::4:1$
Div. the first couplet by $x-y$, $x+y:x-y::4:1$
Adding and subtracting terms, $2x:2y::5:3$
Div. the first couplet by 2, $x:y::5:3$
Multiplying extremes and means, $3x=5y$ (2)
Dividing (1) by (2), $\frac{y}{3}=\frac{27}{y}$, $y^2=81$
Therefore, $y=9$, and $x=15$.

Ex. 11. Let $x=$ one number, and $y=$ the other.
By the conditions, $x-y : x+y :: 2 : 3$ (1)
And $x-y : xy :: 2 : 5$, or $5x-5y=2xy$ (2)
Adding and subtracting terms in (1), $2x : 2y :: 5 : 1$
Div. first couplet, and mult. extremes and means,
$x : y :: 5 : 1$, and $x=5y$
Substituting in (2), $25y-5y=10y^2$
Therefore, $20y=10y^2$, $y=2$, and $x=10$.

Ex. 12. Let $x=$ the greater, and $y=$ the less.
By the conditions, $x+y=24$ (1)
And $xy : x^2+y^2 :: 3 : 10$
Mult. antecedents by 2, and adding terms,
$2xy : x^2+2xy+y^2 :: 6 : 16$ (2)
Squaring (1), $x^2+2xy+y^2=576$ (3)
Substituting in (2), $2xy : 576 :: 6 : 16$ (4)
Converting (4) into an equation, $4xy=432$ (5)
Subtracting (5) from (3), $x^2-2xy+y^2=144$
Extract. the square root, $x-y=12$, $x=18$, and $y=6$.

Another Solution. Let $x=$ the greater, $24-x=$ the less.
By the conditions, $x(24-x) : x^2+(24-x)^2 :: 3 : 10$
Expanding, $24x-x^2 : x^2+576-48x+x^2 :: 3 : 10$
Mult. antecedents by 2, $48x-2x^2 : 2x^2-48x+576 :: 6 : 10$
Add. and subtract. terms, $576 : 4x^2-96x+576 :: 16 : 4$
Extract. the square root, $24 : 2x-24 :: 4 : 2$
Div. antecedents by 4, and consequents by 2,
$6 : x-12 :: 1 : 1$
Mult. extremes and means, $x-12=6$, and $x=18$.

Ex. 13. Let $2x$ and $3x=$ the numbers.
By the conditions, $3x+6 : 2x-6 :: 3 : 1$
Adding and subtracting terms, $x+12 : 5x :: 2 : 4$
Multiplying extremes and means, $4x+48=10x$, and $x=8$
Therefore, $3x=24$, and $2x=16$.

Ex. 14. Let $x =$ the half sum, and $y =$ the half difference: then $x+y =$ the greater, and $x-y =$ the less.

By the conditions, $x^2-y^2=320$

And $6x^2y+2y^3 : 8y^3 :: 61 : 1$

Multiplying antecedents by 4, $24x^2y+8y^3 : 8y^3 :: 244 : 1$

Div. first couplet by $y$, and subtract. terms, $24x^2 : 8y^2 :: 243 : 1$

Div. first couplet by 8, and anteced. by 3, $x^2 : y^2 :: 81 : 1$

Extracting, $x : y :: 9 : 1$, or $x=9y$

Therefore, $81y^2-y^2=320$, $y^2=4$, and $y=2$

Hence, $x=18$, $x+y=20$, $x-y=16$.

ANOTHER SOLUTION. Let $x$ and $y$ be the two numbers.

By the conditions, $xy=320$ (1)

And $x^3-y^3 : (x-y)^3 :: 61 : 1$

Or, $x^3-y^3 : x^3-3x^2y+3xy^2-y^3 :: 61 : 1$

Subtracting terms, $3x^2y-3xy^2 : (x-y)^3 :: 60 : 1$

Div. first couplet by $x-y$, $3xy : (x-y)^2 :: 60 : 1$

Or, as $3xy=3\times320=960$, $960 : (x-y)^2 :: 60 : 1$

Div. antecedents by 15, $64 : (x-y)^2 :: 4 : 1$

Extract. the square root, $8 : x-y :: 2 : 1$, or $8=2x-2y$

Hence, $x-y=4$, and $(x-y)^2=16$ (2)

Mult. (1) by 4, $4xy=1280$ (3)

Add. (2) and (3), $x^2+2xy+y^2=1296$, or $x+y=36$

Therefore, $x=20$, and $y=16$.

Ex. 15. Let $x$ and $y$ be the numbers.

By the conditions, $x : y :: 16 : 9$ (1)

And $x : 24 :: 24 : y$

By alternation in (1), $x : 16 :: y : 9$

Mult. conseq. by 3, and div. them by 2, $x : 24 :: y : 13\frac{1}{2}$

By equality of ratios, $y : 13\frac{1}{2} :: 24 : y$

Hence, $y^2=324$, and $y=18$

Substituting in (1), $x : 18 :: 16 : 9$

Or, $x : 2 :: 16 : 1$, and $x=32$.

## ARITHMETICAL PROGRESSION.

### Problems under Article 436.

PROB. 1. $a=7$, $d=3$, $n=9$: required to find $z$.

By the formula, $z=a+(n-1)\times d$

Subst. of num., $z=7+(9-1)\times 3=7+8\times 3=7+24=31$.

PROB. 2. $z=60$, $n=12$, $d=5$: to find $a$.

By the formula, $a=z-(n-1)\times d$

Subst. $a=60-(12-1)\times 5=60-11\times 5=60-55=5$.

PROB. 3. $a=1$, $z=43$, $n=8$: to find $d$.

By the formula, $d=\dfrac{z-a}{n-1}$

Substituting, $d=\dfrac{43-1}{8-1}=\dfrac{42}{7}=6$

Therefore the series is 1, 7, 13, 19, 25, 31, 37, 43.

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### Examples under Article 439.

Ex. 1. $a=3$, $d=2$, $n=20$: to find $s$.

By the formula, $s=\dfrac{2a+(n-1)\times d}{2}\times n$

Substituting, $s=\dfrac{2\times 3+(20-1)\times 2}{2}\times 20$

$$=\frac{6+38}{2}\times 20=440=\text{sum of the series.}$$

Ex. 2. In going from the box to the first stone, and in returning, two yards were travelled over; and in going from the box to the second stone, and in returning, four yards were travelled over. Therefore $2=$ the first term, and $2=$ the common difference.

By the conditions, $a=2$, $d=2$, $n=100$: to find $s$.

By the formula, $$s=\frac{2a+(n-1)\times d}{2}\times n$$

Substituting, $$s=\frac{2\times 2+(100-1)\times 2}{2}\times 100$$

Reducing, $$s=\frac{4+198}{2}\times 100=\frac{202}{2}\times 100=10100$$

10100 yards=5 miles and 1300 yards.

Ex. 3. $a=\frac{1}{3}$, $d=\frac{1}{3}$, $n=150$: to find $s$.

By the formula, $$s=\frac{2a+(n-1)\times d}{2}\times n$$

Substituting, $$s=\frac{\frac{1}{3}\times 2+(150-1)\times \frac{1}{3}}{2}\times 150$$

Reducing, $$s=\frac{\frac{2}{3}+149\times \frac{1}{3}}{2}\times 150=\frac{\frac{151}{3}}{2}\times 150=3775.$$

Ex. 4. $a=5$, $n=30$, $s=1455$: to find $d$.

By the formula, $$d=\frac{2s-2an}{n^2-n}$$

Substituting, $$d=\frac{2\times 1455-2\times 5\times 30}{(30)^2-30}$$

Reducing, $$d=\frac{2910-300}{900-30}=\frac{2610}{870}=3.$$

Ex. 5. $a=7$, $d=2$, $s=567$: to find $n$.

By the formula, $$n=\frac{\sqrt{8ds+(2a-d)^2}-2a+d}{2d}$$

Substituting, $$n=\frac{\sqrt{8\times 2\times 567+(14-2)^2}-2\times 7+2}{2\times 2}$$

Reducing, $$n=\frac{\sqrt{9072+144}-14+2}{4}$$

Or, $$n=\frac{\sqrt{9216}-12}{4}=\frac{96-12}{4}=21.$$

Ex. 6. $a=1$, $d=\frac{1}{2}$, $n=32$: to find $s$.

By the formula, $s=\dfrac{2a+(n-1)\times d}{2}\times n$

Substituting, $s=\dfrac{2+(32-1)\times\frac{1}{2}}{2}\times 32$

Reducing, $s=\dfrac{2+31\times\frac{1}{2}}{2}\times 32=\dfrac{35}{4}\times 32=280.$

Ex. 7. $a=10$, $d=20$, $n=47$: to find $s$.

Subst. in same formula, $s=\dfrac{2\times 10+(47-1)\times 20}{2}\times 47$

Reducing, $s=\dfrac{20+46\times 20}{2}\times 47=\dfrac{20+920}{2}\times 47$

Or, $s=470\times 47=22090=\$220.90.$

Ex. 8. $a=1$, $d=1$, $n=365$: to find $s$.

Subst. in same formula, $s=\dfrac{2+(365-1)\times 1}{2}\times 365$

Reducing, $s=\dfrac{2+364}{2}\times 365=66795=\$667.95.$

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**Problems under Article 443.**

Prob. 1. Let $x=$ the second number, and $y=$ the common difference.

The series will be $x-y$, $x$, $x+y$, $x+2y$.

By the conditions, $(x-y)+x+(x+y)+(x+2y)=56$
And $(x-y)^2+x^2+(x+y)^2+(x+2y)^2=864$
Or by reduction, $4x+2y=56$ (1)
And $4x^2+4xy+6y^2=864$ (2)
Div. (1) by 2, and squaring, $4x^2+4xy+y^2=784$ (3)
Subtracting (3) from (2), $5y^2=80$, and $y^2=16$
Therefore, $y=4$, and $x=12$

**And the series will be 8, 12, 16, 20.**

**Prob.** 2. Let $x=$ the second number, and $y=$ the common difference.

The series will be $x-y,\ x,\ x+y$.

By the conditions, $(x-y)+x+(x+y)=9$ (1)
And $(x-y)^3+x^3+(x+y)^3=153$ (2)
Reducing (1), $3x=9$, and $x=3$
Reducing (2), $3x^3+6xy^2=153$
Substituting, $81+18y^2=153$, or $18y^2=72$
Therefore, $y^2=4$, and $y=2$.

The series will be 1, 2, 3.

**Prob.** 3. Let $x=$ the second number, and $y=$ the common difference.

The series will be $x-y,\ x,\ x+y$.

By the conditions, $(x-y)+x+(x+y)=15$ (1)
And $(x-y)^2+(x+y)^2=58$ (2)
Reducing (1), $3x=15$, and $x=5$
Reducing (2), $2x^2+2y^2=58$, or $x^2+y^2=29$
Substituting, $25+y^2=29$, $y^2=4$, and $y=2$.

The series will be 3, 5, 7.

**Prob.** 4. Let $x-3y=$ the first term, and $2y=$ the common difference.

The series will be $x-3y,\ x-y,\ x+y,\ x+3y$.

By the conditions, $(x-3y)^2+(x-y)^2=34$ (1)
And $(x+y)^2+(x+3y)^2=130$ (2)
Reducing (1), $2x^2-8xy+10y^2=34$ (3)
Reducing (2), $2x^2+8xy+10y^2=130$ (4)
Subtract. (3) from (4), $16xy=96$, $xy=6$, or $y=\frac{6}{x}$
Adding (3) to (4), $4x^2+20y^2=164$, or $x^2+5y^2=41$
Or, $5y^2=41-x^2$, and $y=\sqrt{\frac{41-x^2}{5}}$

Equating the values of $y$, $\sqrt{\frac{41-x^2}{5}}=\frac{6}{x}$

Squaring each side, $\frac{41-x^2}{5}=\frac{36}{x^2}$, or $41x^2-x^4=180$

Changing the signs, $x^4-41x^2=-180$

Completing the square, $4x^4-164x^2+1681=1681-720$

Extracting the square root, $2x^2-41=\sqrt{961}=31$

Or, $2x^2=72$, $x^2=36$, and $x=6$

Hence $y=\frac{6}{x}=\frac{6}{6}=1.$

And the series will be 3, 5, 7, 9.

Another Solution. Let $x$ = the second term, and $y$ = the common difference.

Then the series will be $x-y$, $x$, $x+y$, $x+2y$.

By the conditions, $(x-y)^2+x^2=34$ (1)

And $(x+y)^2+(x+2y)^2=130$ (2)

Reducing (1), $2x^2-2xy+y^2=34$ (3)

Reducing (2), $2x^2+6xy+5y^2=130$ (4)

Sub. (3) from (4), $8xy+4y^2=96$, or $y^2+2xy=24$ (5)

Mult. (3) by 3, $6x^2-6xy+3y^2=102$ (6)

Adding (6) to (4), $8x^2+8y^2=232$, or $x^2+y^2=29$

Or, $y^2=29-x^2$, and $y=\sqrt{29-x^2}$

Subst. these values in (5), $29-x^2+2x\sqrt{29-x^2}=24$

Or, $2x\sqrt{29-x^2}=x^2-5$

Squaring each side, $4x^2(29-x^2)=x^4-10x^2+25$

Reducing, $5x^4-126x^2=-25$

Comp. the square, $100x^4-2520x^2+15876=15876-500$

Extract. the root, $10x^2-126=124$, $10x^2=250$

Hence, $x^2=25$, $x=5$, and $y=\sqrt{29-25}=2.$

And the series will be 3, 5, 7, 9.

**Prob.** 5. Let the digits be equal to $x-y$, $x$, and $x+y$ respectively.

Then the number $=100(x-y)+10x+(x+y)=111x-99y$.

By the conditions, $\dfrac{111x-99y}{3x}=26$ (1)

And $111x-99y+198=100(x+y)+10x+(x-y)$ (2)

Reducing (2), $111x-99y+198=111x+99y$

Or, $198y=198$, and $y=1$

Subst. the value of $y$ in (1), $\dfrac{111x-99}{3x}=26$

Reducing, $111x-99=78x$, and $x=3$.

The digits are 2, 3, 4; and the number is 234.

**Prob.** 6. Let the series be $x-3y$, $x-y$, $x+y$, $x+3y$.

By the conditions, $(x-3y)^2+(x+3y)^2=200$ (1)

And $(x-y)^2+(x+y)^2=136$ (2)

Reducing (1), $2x^2+18y^2=200$ (3)

Reducing (2), $2x^2+2y^2=136$ (4)

Subtracting (4) from (3), $16y^2=64$, $y^2=4$, $y=2$

Subst. this value in (4), $2x^2+8=136$

Reducing, $2x^2=128$, $x^2=64$, $x=8$

And the series is 2, 6, 10, 14.

**Prob.** 7. Let the series be $x-3y$, $x-y$, $x+y$, $x+3y$.

By the cond. $(x-3y)+(x-y)+(x+y)+(x+3y)=28$ (1)

And $(x-3y)\times(x-y)\times(x+y)\times(x+3y)=585$ (2)

Reducing (1), $4x=28$, and $x=7$

Subst. in (2), and reducing, $2401-490y^2+9y^4=585$

Or, $9y^4-490y^2=-1816$

Completing the square, $324y^4-17640y^2+240100=174724$

Extract. the root, $18y^2-490=\pm 418$, $18y^2=72$, $y=2$

And the numbers are 1, 5, 9, 13.

# GEOMETRICAL PROGRESSION.

### Problems under Article 450.

**Prob.** 1. $a=4$, $z=256$, $n=4$: to find $r$.

By the formula, $$r=\left(\frac{z}{a}\right)^{\frac{1}{n-1}}$$

Substituting, $$r=\left(\frac{256}{4}\right)^{\frac{1}{4-1}}=(64)^{\frac{1}{3}}=4$$

And the series is 4, 16, 64, 256.

**Prob.** 2. $a=\frac{1}{9}$, $z=9$, $n=5$: to find $r$.

By the formula, $$r=\left(\frac{z}{a}\right)^{\frac{1}{n-1}}$$

Substituting, $$r=\left(\frac{9}{\frac{1}{9}}\right)^{\frac{1}{5-1}}=(81)^{\frac{1}{4}}=3.$$

The three means are $\frac{1}{3}$, 1, 3; and the series, $\frac{1}{9}$, $\frac{1}{3}$, 1, 3, 9.

### Problems under Article 452.

**Prob.** 1. $a=6$, $z=1458$, $r=3$: to find $s$.

By the formula, $$s=\frac{rz-a}{r-1}$$

Substituting, $$s=\frac{3\times1458-6}{3-1}=\frac{4368}{2}=2184.$$

**Prob.** 2. $a=\frac{1}{2}$, $r=\frac{1}{3}$, $n=5$: to find $s$.

By the formula, $$s=\frac{ar^n-a}{r-1}$$

Substituting, $$s=\frac{\frac{1}{2}\times(\frac{1}{3})^5-\frac{1}{2}}{\frac{1}{3}-1}=\frac{\frac{1}{2}\times\frac{1}{243}-\frac{1}{2}}{-\frac{2}{3}}$$

Or, $$s=\frac{\frac{1}{486}-\frac{1}{2}}{-\frac{2}{3}}=\frac{-\frac{242}{486}}{-\frac{2}{3}}=\frac{242\times3}{486\times2}=\frac{121}{162}.$$

PROB. 3. $a=1$, $r=3$, $n=12$: to find $s$.

$$ar^n=1\times3^{12}=1\times531441$$

By the formula, $s=\dfrac{ar^n-a}{r-1}$

Substituting, $s=\dfrac{531441-1}{3-1}=\dfrac{531440}{2}=265720.$

PROB. 4. $a=1$, $r=\frac{2}{3}$, $n=10$.

Then $ar^n=1\times(\frac{2}{3})^{10}=\dfrac{1024}{59049}$

By the formula, $s=\dfrac{ar^n-a}{r-1}$

Substituting, $s=\dfrac{\frac{1024}{59049}-1}{\frac{2}{3}-1}=\dfrac{58025}{59049}\times\dfrac{3}{1}=\dfrac{174075}{59049}.$

### Problems under Article 454.

PROB. 1. Let the three numbers be $x$, $y$, and $z$.

By the conditions, $x:y::y:z$, or $xz=y^2$ (1)

And $x+y+z=14$ (2)

And $x^2+y^2+z^2=84$ (3)

Subst. for $y^2$ in (3) its value from (1), $x^2+xz+z^2=84$ (4)

Adding (1) to (4), $x^2+2xz+z^2=84+y^2$

Extracting the root, $x+z=\sqrt{84+y^2}$

Transposing $y$ in (2), $x+z=14-y$ (5)

Equating the values of $(x+z)$, $\sqrt{84+y^2}=14-y$

Squaring each side, $84+y^2=196-28y+y^2$

Reducing, $28y=-84+196=112$, and $y=4$

Subst. the value of $y$ in (5), and squaring, $(x+z)^2=100$ (6)

Subst. the value of $y$ in (1), and mult. by 4, $4xz=64$ (7)

Subtracting (7) from (6), $x^2-2xz+z^2=36$

Extracting the root, $x-z=6$

Therefore, $2x=16$ and $x=8$, $2z=4$ and $z=2$.

And the numbers are 2, 4, 8.

**Prob. 2.** Let $x=$ the first term, $y=$ the common ratio.

The series will be $x$, $xy$, $xy^2$.

By the conditions, $x\times xy\times xy^2=x^3y^3=64$ (1)

And $x^3+x^3y^3+x^3y^6=584$ (2)

Dividing (2) by (1), $\frac{1}{y^3}+1+y^3=\frac{584}{64}=\frac{73}{8}$

Clearing of fractions, $8+8y^3+8y^6=73y^3$

Reducing, $y^6-\frac{65}{8}y^3=-1$

Completing the square, $y^6-\frac{65}{8}y^3+\frac{4225}{256}=\frac{4225}{256}-1$

Extract. the root, $y^3=\frac{65}{16}\pm\sqrt{\frac{4225-256}{256}}=\frac{65}{16}\pm\sqrt{\frac{3969}{256}}$

Or, $y^3=\frac{65}{16}\pm\frac{63}{16}=\frac{128}{16}=\frac{64}{8}$

Hence, $y=\sqrt[3]{\frac{64}{8}}=\frac{4}{2}=2$, and $x=\sqrt{\frac{64}{8}}=\frac{4}{2}=2$.

And the required numbers are 2, 4, and 8.

**Prob. 3.** Let $x$, $xy$, and $xy^2$ be the three numbers.

By the conditions, $x+xy^2=52$ (1)

And $x^2y^2=100$ (2)

Extracting the square root of (2), $xy=10$ (3)

Dividing (1) by (3), $\frac{1}{y}+y=\frac{52}{10}$

Clearing of fractions, $10+10y^2=52y$

Reducing, $y^2-\frac{52}{10}y=-1$

Completing the square, $y^2-\frac{52}{10}y+\left(\frac{26}{10}\right)^2=\left(\frac{26}{10}\right)^2-1$

Extracting the root, $y=\frac{26}{10}\pm\sqrt{\frac{676-100}{100}}=\frac{26}{10}\pm\sqrt{\frac{576}{100}}$

Or, $y=\frac{26}{10}\pm\frac{24}{10}=5$, and $x=\frac{10}{5}=2$.

And the required numbers are 2, 10, and 50.

PROB. 4. Let $x$, $xy$, $xy^2$, and $xy^3$ be the numbers.

By the conditions, $x+xy=15$ (1)

And $xy^2+xy^3=60$ (2)

Dividing (2) by (1), $y^2=4$, and $y=2$

Substituting in (1), $x+2x=15$, $3x=15$, and $x=5$.

And the required numbers are 5, 10, 20, and 40.

PROB. 5. Let $xy^2$, $xy$, and $x$ represent the series.

By the conditions, $x+xy+xy^2=210$ (1)

And $xy^2=x+90$ (2)

Subst. in (1) the value of $xy^2$, $x+xy+x+90=210$

Reducing, $2x+xy=120$

Dividing by $(2+y)$, $x=\dfrac{120}{2+y}$

Trans. $x$ in (2) and div. by $(y^2-1)$, $x=\dfrac{90}{y^2-1}$

Equating the values of $x$, $\dfrac{90}{y^2-1}=\dfrac{120}{2+y}$

Clearing of fractions, $180+90y=120y^2-120$

Dividing by 30, $6+3y=4y^2-4$

Reducing, $y^2-\dfrac{3}{4}y=\dfrac{10}{4}$

Completing the square, $y^2-\dfrac{3}{4}y+\left(\dfrac{3}{8}\right)^2=\dfrac{10}{4}+\left(\dfrac{3}{8}\right)^2$

Extracting the root, $y=\dfrac{3}{8}\pm\sqrt{\dfrac{9}{64}+\dfrac{10}{4}}=\dfrac{3}{8}\pm\sqrt{\dfrac{169}{64}}$

Or, $y=\dfrac{3}{8}\pm\dfrac{13}{8}=2$, and $x=\dfrac{120}{2+y}=\dfrac{120}{4}=30$.

And the portions of the servants are \$30, \$60, and \$120.

PROB. 6. Let $x$, $xy$, and $xy^2$ be the series.

By the conditions, $xy^2=x+15$ (1)

And $x^2y^4-x^2 : x^2+x^2y^2+x^2y^4 :: 5 : 7$ (2)

Div. first couplet by $x^2$, $y^4-1 : 1+y^2+y^4 :: 5 : 7$

Subt. anteced. from conseq., $y^4-1 : y^2+2 :: 5 : 2$

Mult. extremes and means, $2y^4-2=5y^2+10$

Reducing, $y^4-\frac{5}{2}y^2=6$

Completing the square, $y^4-\frac{5}{2}y^2+\left(\frac{5}{4}\right)^2=6+\left(\frac{5}{4}\right)^2$

Extracting the root, $y^2=\frac{5}{4}\pm\sqrt{6+\frac{25}{16}}=\frac{5}{4}\pm\sqrt{\frac{121}{16}}$

Or, $y^2=\frac{5}{4}\pm\frac{11}{4}=\frac{16}{4}=4$, and $y=2$

Subst. the value of $y$ in (1), $4x=x+15$, $3x=15$, $x=5$.

And the series is 5, 10, 20.

PROB. 7. Let $x$, $xy$, $xy^2$, and $xy^3$ be the series.

By the conditions, $xy^3-xy=24$ (1)

And $x+xy^3 : xy+xy^2 :: 7 : 3$ (2)

Div. first couplet by $(x+xy)$, $y^2-y+1 : y :: 7 : 3$

Mult. extremes and means, $3y^2-3y+3=7y$

Reducing, $y^2-\frac{10}{3}y=-1$

Completing the square, $y^2-\frac{10}{3}y+\left(\frac{5}{3}\right)^2=\left(\frac{5}{3}\right)^2-1$

Extracting the root, $y=\frac{5}{3}\pm\sqrt{\frac{25-9}{9}}=\frac{5}{3}\pm\sqrt{\frac{16}{9}}$

Or, $y=\frac{5}{3}\pm\frac{4}{3}=3$

Subst. the value of $y$ in (1), $27x-3x=24$, $24x=24$, $x=1$.

And the required numbers are 1, 3, 9, 27.

◆

## GREATEST COMMON MEASURE.

Ex. 2.

$$\begin{array}{l|l} x^2+2bx+b^2) & x^3-\;b^2x\,(x \\ & \underline{x^3+2bx^2+b^2x} \\ & *\;-2bx^2-2b^2x \end{array}$$

Dividing this remainder by $-2bx$ (Art. 477), we have, as a new divisor,

$$\begin{array}{l}
x+b)\,x^2+2bx+b^2\,(x+b \\
\qquad\ \ \underline{x^2+\ bx\qquad\ } \\
\qquad\ \ *\quad\ bx+b^2 \\
\qquad\qquad\ \ \underline{bx+b^2} \qquad x+b=\text{G. C. M.}
\end{array}$$

Ex. 3. Dividing $(a^2c+a^2x)$ by $a^2$ (Art. 477), we have, as a divisor, $(c+x)$.

$$\begin{array}{l}
c+x)\,cx+x^2\,(x \\
\qquad\ \ \underline{cx+x^2} \qquad c+x=\text{G. C. M.} \\
\qquad\qquad *
\end{array}$$

Ex. 4. By dividing one of the quantities by 2, and the other by 3, each becomes $(x^2-8x-3)$, which is the greatest common measure.

Ex. 5. Dividing $(a^5-b^2a^3)$ by $a^3$, we have, as a divisor, $(a^2-b^2)$.

$$\begin{array}{l}
a^2-b^2)\,a^4-b^4\,(a^2-b^2 \\
\qquad\quad\ \ \underline{a^4-a^2b^2\quad\ } \\
\qquad\quad\ \ *\quad\ a^2b^2-b^4 \\
\qquad\qquad\quad\ \ \underline{a^2b^2-b^4} \qquad a^2-b^2=\text{G. C. M.}
\end{array}$$

Ex. 6. Dividing $(xy+y)$ by $y$, we have, as a divisor, $(x+1)$.

$$\begin{array}{l}
x+1)\,x^2-1\,(x-1 \\
\qquad\ \ \underline{x^2+x\ \ } \\
\qquad\ \ *\ -x-1 \\
\qquad\quad\ \ \underline{-x-1} \qquad (x+1)=\text{G. C. M.}
\end{array}$$

Ex. 7. Dividing $(x^4-a^4)$ by $(x^2+a^2)$, we have, as a divisor,

$$\begin{array}{l}
x^2-a^2)\,x^3-a^3\,(x \\
\qquad\quad\ \ \underline{x^3-a^2x\quad\ } \\
\qquad\quad\ \ *\quad\ a^2x-a^3
\end{array}$$

Dividing this remainder by $a^2$, we have, as a new divisor,

$$\begin{array}{l}
x-a)\,x^2-a^2\,(x+a \\
\qquad\ \ \underline{x^2-ax\qquad\ } \\
\qquad\ \ *\quad\ ax-a^2 \\
\qquad\qquad\ \underline{ax-a^2} \qquad x-a=\text{G. C. M.}
\end{array}$$

Ex. 8.
$$\begin{array}{rl}
a^2-3ab+2b^2) & a^2-\;ab-2b^2\,(1 \\
 & \underline{a^2-3ab+2b^2} \\
 & *\quad 2ab-4b^2
\end{array}$$

Dividing this remainder by $2b$,
$$\begin{array}{rl}
a-2b) & a^2-3ab+2b^2\,(a-b \\
 & \underline{a^2-2ab\qquad\;} \\
 & *-\;ab+2b^2 \\
 & \underline{\;\;-\;ab+2b^2}
\end{array}\qquad a-2b=\text{G. C. M.}$$

Ex. 9.
$$\begin{array}{rl}
a^3-a^2x-ax^2+x^3) & a^4-x^4\,(a \\
 & \underline{a^4-a^3x-a^2x^2+ax^3} \\
 & *\quad a^3x+a^2x^2-ax^3-x^4
\end{array}$$

Dividing this remainder by $(ax+x^2)$,
$$\begin{array}{rl}
ax+x^2) & a^3x+a^2x^2-ax^3-x^4\,(a^2-x^2 \\
 & \underline{a^3x+a^2x^2\qquad\qquad\;} \\
 & *\quad\;\; *\;\; -ax^3-x^4 \\
 & \qquad\qquad\underline{-ax^3-x^4}
\end{array}$$

Taking $(a^2-x^2)$ as a new divisor,
$$\begin{array}{rl}
a^2-x^2) & a^3-a^2x-ax^2+x^3\,(a-x \\
 & \underline{a^3\qquad\;\; -ax^2\quad\;\;} \\
 & *-a^2x\;\;*\;\;+x^3 \\
 & \underline{\;\;-a^2x\qquad+x^3}
\end{array}\qquad a^2-x^2=\text{G. C. M.}$$

Ex. 10. $(a^3-ab^2)=a\times(a^2-b^2)=a\times(a-b)\times(a+b)$.
The first two factors are evidently not contained in the second quantity, and may therefore be omitted.

Dividing by $(a+b)$,
$$\begin{array}{rl}
a+b) & a^2+2ab+b^2\,(a+b \\
 & \underline{a^2+\;ab\qquad\;} \\
 & *\quad\;\; ab+b^2 \\
 & \qquad\;\underline{ab+b^2}
\end{array}\qquad (a+b)=\text{G. C. M.}$$

The examples under Articles 482 and 483 are so simple, that it is thought unnecessary to give solutions of them here.

## INVOLUTION OF BINOMIALS.

In the expansion of quantities under this head, the letters $A$, $B$, $C$, $D$, &c., are used to denote the *undetermined* coefficients.

### Example 2.

$$(1+x)^{\frac{1}{2}}=1^{\frac{1}{2}}+A1^{-\frac{1}{2}}x+B1^{-\frac{3}{2}}x^2+C1^{-\frac{5}{2}}x^3+D1^{-\frac{7}{2}}x^4,\ \text{\&c.}$$

$$A=\frac{1}{2}.\qquad B=\frac{1}{2}\times\frac{-\frac{1}{2}}{2}=\frac{1}{2}\times-\frac{1}{4}=-\frac{1}{2.4}.$$

$$C=-\frac{1}{2.4}\times\frac{-\frac{3}{2}}{3}=-\frac{1}{2.4}\times-\frac{3}{6}=\frac{3}{2.4.6}.$$

$$D=\frac{3}{2.4.6}\times\frac{-\frac{5}{2}}{4}=\frac{3}{2.4.6}\times-\frac{5}{8}=-\frac{3.5}{2.4.6.8}.$$

Disregarding the 1, as all its powers and roots are 1, the series will be,

$$(1+x)^{\frac{1}{2}}=1+\frac{x}{2}-\frac{x^2}{2.4}+\frac{3x^3}{2.4.6}-\frac{3.5x^4}{2.4.6.8},\ \text{\&c.}$$

### Example 3.

$$(1+1)^{\frac{1}{2}}=1^{\frac{1}{2}}+A1^{-\frac{1}{2}}+B1^{-\frac{3}{2}}+C1^{-\frac{5}{2}}+D1^{-\frac{7}{2}}+E1^{-\frac{9}{2}},\ \text{\&c.}$$

$$A=\frac{1}{2}.\qquad B=\frac{1}{2}\times\frac{-\frac{1}{2}}{2}=\frac{1}{2}\times-\frac{1}{4}=-\frac{1}{2.4}.$$

$$C=-\frac{1}{2.4}\times\frac{-\frac{3}{2}}{3}=-\frac{1}{2.4}\times-\frac{3}{6}=\frac{3}{2.4.6}.$$

$$D=\frac{3}{2.4.6}\times\frac{-\frac{5}{2}}{4}=\frac{3}{2.4.6}\times-\frac{5}{8}=-\frac{3.5}{2.4.6.8}.$$

$$E=-\frac{3.5}{2.4.6.8}\times\frac{-\frac{7}{2}}{5}=-\frac{3.5}{2.4.6.8}\times-\frac{7}{10}=\frac{3.5.7}{2.4.6.8.10}.$$

Series of

$$(1+1)^{\frac{1}{2}}=1+\frac{1}{2}-\frac{1}{2.4}+\frac{3}{2.4.6}-\frac{3.5}{2.4.6.8}+\frac{3.5.7}{2.4.6.8.10},\ \text{\&c.}$$

### EXAMPLE 4.

$$a^{\frac{1}{2}}\times\left(1+\frac{x}{a}\right)^{\frac{1}{2}}=$$

$$a^{\frac{1}{2}}\times\left(1^{\frac{1}{2}}+A1^{-\frac{1}{2}}\frac{x}{a}+B1^{-\frac{3}{2}}\frac{x^2}{a^2}+C1^{-\frac{5}{2}}\frac{x^3}{a^3}+D1^{-\frac{7}{2}}\frac{x^4}{a^4}, \&c.\right)$$

$$A=\frac{1}{2}. \qquad B=\frac{1}{2}\times\frac{-\frac{1}{2}}{2}=\frac{1}{2}\times-\frac{1}{4}=-\frac{1}{2.4}.$$

$$C=-\frac{1}{2.4}\times\frac{-\frac{3}{2}}{3}=-\frac{1}{2.4}\times-\frac{3}{6}=\frac{3}{2.4.6}.$$

$$D=\frac{3}{2.4.6}\times\frac{-\frac{5}{2}}{4}=\frac{3}{2.4.6}\times-\frac{5}{8}=-\frac{3.5}{2.4.6.8}.$$

$$a^{\frac{1}{2}}\times\left(1+\frac{x}{a}\right)^{\frac{1}{2}}=a^{\frac{1}{2}}\times\left(1+\frac{x}{2a}-\frac{x^2}{2.4a^2}+\frac{3x^3}{2.4.6a^3}-\frac{3.5x^4}{2.4.6.8a^4}, \&c.\right)$$

### EXAMPLE 5.

$$a^{\frac{1}{3}}\times\left(1+\frac{b}{a}\right)^{\frac{1}{3}}=$$

$$a^{\frac{1}{3}}\left(1^{\frac{1}{3}}+A1^{-\frac{2}{3}}\frac{b}{a}+B1^{-\frac{5}{3}}\frac{b^2}{a^2}+C1^{-\frac{8}{3}}\frac{b^3}{a^3}+D1^{-\frac{11}{3}}\frac{b^4}{a^4}, \&c.\right)$$

$$A=\frac{1}{3}. \qquad B=\frac{1}{3}\times\frac{-\frac{2}{3}}{2}=\frac{1}{3}\times-\frac{2}{6}=-\frac{2}{3.6}.$$

$$C=-\frac{2}{3.6}\times\frac{-\frac{5}{3}}{3}=-\frac{2}{3.6}\times-\frac{5}{9}=\frac{2.5}{3.6.9}.$$

$$D=\frac{2.5}{3.6.9}\times\frac{-\frac{8}{3}}{4}=\frac{2.5}{3.6.9}\times-\frac{8}{12}=-\frac{2.5.8}{3.6.9.12}.$$

$$a^{\frac{1}{3}}\times\left(1+\frac{b}{a}\right)^{\frac{1}{3}}=$$

$$a^{\frac{1}{3}}\times\left(1+\frac{b}{3a}-\frac{2b^2}{3.6a^2}+\frac{2.5b^3}{3.6.9a^3}-\frac{2.5.8b^4}{3.6.9.12a^4}, \&c.\right)$$

EXAMPLE. 6. $a^{\frac{1}{4}}\times\left(1-\frac{b}{a}\right)^{\frac{1}{4}}=$

$$a^{\frac{1}{4}}\times\left(1^{\frac{1}{4}}-A1^{-\frac{3}{4}}\frac{b}{a}+B1^{-\frac{7}{4}}\frac{b^2}{a^2}-C1^{-\frac{11}{4}}\frac{b^3}{a^3}+D1^{-\frac{15}{4}}\frac{b^4}{a^4},\ \&c.\right)$$

$$A=\frac{1}{4}.\qquad B=\frac{1}{4}\times\frac{-\frac{3}{4}}{2}=\frac{1}{4}\times-\frac{3}{8}=-\frac{3}{4.8}.$$

$$C=-\frac{3}{4.8}\times\frac{-\frac{7}{4}}{3}=-\frac{3}{4.8}\times-\frac{7}{12}=\frac{3.7}{4.8.12}.$$

$$D=\frac{3.7}{4.8.12}\times\frac{-\frac{11}{4}}{4}=\frac{3.7}{4.8.12}\times-\frac{11}{16}=-\frac{3.7.11}{4.8.12.16}.$$

As the terms which have a *plus* coefficient, also have a *minus* sign before them; consequently the signs of all except the first term will be *minus*.

$$a^{\frac{1}{4}}\times\left(1-\frac{b}{a}\right)^{\frac{1}{4}}=$$

$$a^{\frac{1}{4}}\times\left(1-\frac{b}{4a}-\frac{3b^2}{4.8a^2}-\frac{3.7b^3}{4.8.12a^3}-\frac{3.7.11b^4}{4.8.12.16a^4},\ \&c.\right)$$

EXAMPLE 7. $(a+x)^{-\frac{1}{2}}=a^{-\frac{1}{2}}\times\left(1+\frac{x}{a}\right)^{-\frac{1}{2}}.$

Expanding, we have $a^{-\frac{1}{2}}\times\left(1+\frac{x}{a}\right)^{-\frac{1}{2}}=$

$$a^{-\frac{1}{2}}\times\left(1^{-\frac{1}{2}}+A1^{-\frac{3}{2}}\frac{x}{a}+B1^{-\frac{5}{2}}\frac{x^2}{a^2}+C1^{-\frac{7}{2}}\frac{x^3}{a^3}+D1^{-\frac{9}{2}}\frac{x^4}{a^4},\ \&c.\right)$$

$$A=-\frac{1}{2}.\qquad B=-\frac{1}{2}\times\frac{-\frac{3}{2}}{2}=-\frac{1}{2}\times-\frac{3}{4}=\frac{3}{2.4}.$$

$$C=\frac{3}{2.4}\times\frac{-\frac{5}{2}}{3}=\frac{3}{2.4}\times-\frac{5}{6}=-\frac{3.5}{2.4.6}.$$

$$D=-\frac{3.5}{2.4.6}\times\frac{-\frac{7}{2}}{4}=-\frac{3.5}{2.4.6}\times-\frac{7}{8}=\frac{3.5.7}{2.4.6.8}.$$

$$a^{-\frac{1}{2}}\times\left(1+\frac{x}{a}\right)^{-\frac{1}{2}}=$$

$$a^{-\frac{1}{2}}\times\left(1-\frac{x}{2a}+\frac{3x^2}{2.4a^2}-\frac{3.5x^3}{2.4.6a^3}+\frac{3.5.7x^4}{2.4.6.8a^4},\ \&c.\right)$$

### EXAMPLE 8.

$$(1-x)^{\frac{2}{5}}=1^{\frac{2}{5}}-A1^{-\frac{3}{5}}x+B1^{-\frac{8}{5}}x^2-C1^{-\frac{13}{5}}x^3+D1^{-\frac{18}{5}}x^4, \text{ \&c.}$$

$$A=\frac{2}{5}.\qquad B=\frac{2}{5}\times\frac{-\frac{3}{5}}{2}=\frac{2}{5}\times-\frac{3}{10}=-\frac{2.3}{5.10}.$$

$$C=-\frac{2.3}{5.10}\times\frac{-\frac{8}{5}}{3}=-\frac{2.3}{5.10}\times-\frac{8}{15}=\frac{2.3.8}{5.10.15}.$$

$$D=\frac{2.3.8}{5.10.15}\times\frac{-\frac{13}{5}}{4}=\frac{2.3.8}{5.10.15}\times-\frac{13}{20}=-\frac{2.3.8.13}{5.10.15.20}.$$

$$(1-x)^{\frac{2}{5}}=1-\frac{2x}{5}-\frac{2.3x^2}{5.10}-\frac{2.3.8x^3}{5.10.15}-\frac{2.3.8.13x^4}{5.10.15.20}, \text{ \&c.}$$

### EXAMPLE 9.

$$(1+x)^{-\frac{1}{5}}=1^{-\frac{1}{5}}+A1^{-\frac{6}{5}}x+B1^{-\frac{11}{5}}x^2+C1^{-\frac{16}{5}}x^3, \text{ \&c.}$$

$$A=-\frac{1}{5}.\qquad B=-\frac{1}{5}\times\frac{-\frac{6}{5}}{1}=-\frac{1}{5}\times-\frac{6}{10}=\frac{6}{5.10}.$$

$$C=\frac{6}{5.10}\times\frac{-\frac{11}{5}}{3}=\frac{6}{5.10}\times-\frac{11}{15}=-\frac{6.11}{5.10.15}.$$

$$(1+x)^{-\frac{1}{5}}=1-\frac{x}{5}+\frac{6x^2}{5.10}-\frac{6.11x^3}{5.10.15}, \text{ \&c.}$$

### EXAMPLE 10.

$(a^2+x)^{-\frac{1}{2}}$. Substitute $b$ for $a^2$; expanding, we have,

$$(b+x)^{-\frac{1}{2}}=b^{-\frac{1}{2}}+Ab^{-\frac{3}{2}}x+Bb^{-\frac{5}{2}}x^2+Cb^{-\frac{7}{2}}x^3+Db^{-\frac{9}{2}}x^4, \text{ \&c.}$$

$$A=-\frac{1}{2}.\qquad B=-\frac{1}{2}\times\frac{-\frac{3}{2}}{2}=-\frac{1}{2}\times-\frac{3}{4}=\frac{3}{2.4}.$$

$$C=\frac{3}{2.4}\times\frac{-\frac{5}{2}}{3}=\frac{3}{2.4}\times-\frac{5}{6}=-\frac{3.5}{2.4.6}.$$

$$D=-\frac{3.5}{2.4.6}\times\frac{-\frac{7}{2}}{4}=-\frac{3.5}{2.4.6}\times-\frac{7}{8}=\frac{3.5.7}{2.4.6.8}.$$

Restoring the value of $b$, and using coefficients, we have,

$$(a^2+x)^{-\frac{1}{2}}=\frac{1}{a}-\frac{x}{2a^3}+\frac{3x^2}{2.4a^5}-\frac{3.5x^3}{2.4.6a^7}+\frac{3.5.7x^4}{2.4.6.8a^9}, \text{ \&c.}$$

☞ It should be remembered, that quantities with *negative* indices are most properly and generally expressed as fractions; for instance, $a^{-3}=\frac{1}{a^3}$, or $2b^{-4}=\frac{2}{b^4}$. For further examples, see Art. 207.

---

### Promiscuous Examples.

#### Example 1.

Required the 8th power of $(a+b)$. The terms are as follows:

| | |
|---|---|
| Without coefficients, | $a^8$, $a^7b$, $a^6b^2$, $a^5b^3$, $a^4b^4$, $a^3b^5$, $a^2b^6$, $ab^7$, $b^8$. |
| The coefficients are | 1, 8, 28, 56, 70, 56, 28, 8, 1. |

The terms complete are

$$a^8+8a^7b+28a^6b^2+56a^5b^3+70a^4b^4+56a^3b^5+20a^2b^6+8ab^7+b^8.$$

#### Example 2.

Required the 7th power of $(a-b)$. The terms are as follows:

| | |
|---|---|
| Without coefficients, | $a^7$, $a^6b$, $a^5b^2$, $a^4b^3$, $a^3b^4$, $a^2b^5$, $ab^6$, $b^7$. |
| The coefficients are | 1, 7, 21, 35, 35, 21, 7, 1. |

The terms complete are

$$a^7-7a^6b+21a^5b^2-35a^4b^3+35a^3b^4-21a^2b^5+7ab^6-b^7.$$

#### Example 3.

$$(1-a)^{-1}=1^{-1}-A1^{-2}a+B1^{-3}a^2-C1^{-4}a^3+D1^{-5}a^4-E1^{-6}a^5,\ \&c.$$

$$A=-1.\quad B=-1\times\frac{-2}{2}=1.\quad C=1\times\frac{-3}{3}=-1.$$

$$D=-1\times\frac{-4}{4}=1.\qquad E=1\times\frac{-5}{5}=-1.$$

As the minus coefficients change the minus terms to plus, the series will be,

$$(1-a)^{-1}=1+a+a^2+a^3+a^4+a^5,\ \&c.$$

### EXAMPLE 4.

$$h\times(a-b)^{-1}=h\times(a^{-1}-Aa^{-2}b+Ba^{-3}b^2-Ca^{-4}b^3+Da^{-5}b^4, \text{\&c.})$$

$$A=-1. \qquad B=-1\times\frac{-2}{2}=1.$$

$$C=1\times\frac{-3}{3}=-1. \qquad D=-1\times\frac{-4}{4}=1.$$

$$h(a-b)^{-1}=h\times\left(\frac{1}{a}+\frac{b}{a^2}+\frac{b^2}{a^3}+\frac{b^3}{a^4}+\frac{b^4}{a^5}, \text{\&c.}\right)$$

### EXAMPLE 5.

Let $c$ represent $a^2$; and $d$, $b^2$:

$$(c+d)^{\frac{1}{2}}=c^{\frac{1}{2}}+Ac^{-\frac{1}{2}}d+Bc^{-\frac{3}{2}}d^2+Cc^{-\frac{5}{2}}d^3+Dc^{-\frac{7}{2}}d^4, \text{\&c.}$$

Of course the coefficients are the same as in Examples 1, 2, 3, 4 (Art. 482), and

$$A=\frac{1}{2}. \qquad B=-\frac{1}{2.4}=-\frac{1}{8}.$$

$$C=\frac{3}{2.4.6}=\frac{1}{16}. \qquad D=-\frac{3.5}{2.4.6.8}=-\frac{5}{128}.$$

Restoring the values of $c$ and $d$, viz. $a^2$ and $b^2$, then

$$(a^2+b^2)^{\frac{1}{2}}=a+\frac{b^2}{2a}-\frac{b^4}{8a^3}+\frac{b^6}{16a^5}-\frac{5b^8}{128a^7}, \text{\&c.}$$

### EXAMPLE 6.

$$(a+y)^{-4}=a^{-4}+Aa^{-5}y+Ba^{-6}y^2+Ca^{-7}y^3+Da^{-8}y^4, \text{\&c.}$$

$$A=-4. \qquad B=-4\times\frac{-5}{2}=10.$$

$$C=10\times\frac{-6}{3}=-20. \qquad D=-20\times\frac{-7}{4}=35.$$

$$(a+y)^{-4}=\frac{1}{a^4}-\frac{4y}{a^5}+\frac{10y^2}{a^6}-\frac{20y^3}{a^7}+\frac{35y^4}{a^8}, \text{\&c.}$$

### EXAMPLE 7.

Substituting $a$ for $c^3$ and $b$ for $x^3$, then

$$(a+b)^{\frac{1}{3}}=a^{\frac{1}{3}}+Aa^{-\frac{2}{3}}b+Ba^{-\frac{5}{3}}b^2+Ca^{-\frac{8}{3}}b^3, \text{ \&c.}$$

$$A=\frac{1}{3}. \qquad B=\frac{1}{3}\times\frac{-\frac{2}{3}}{2}=\frac{1}{3}\times-\frac{2}{6}=-\frac{2}{3.6}.$$

$$C=-\frac{2}{3.6}\times\frac{-\frac{5}{3}}{3}=-\frac{2}{3.6}\times-\frac{5}{9}=\frac{2.5}{3.6.9}.$$

Restoring the values of $a$ and $b$, viz. $c^3$ and $x^3$, then

$$(c^3+x^3)^{\frac{1}{3}}=c+\frac{x^3}{3c^2}-\frac{2x^6}{3.6c^5}+\frac{2.5x^9}{3.6.9c^8}, \text{ \&c.}$$

Or bringing $c$ out of the series, and placing it before the vinculum,

$$(c^3+x^3)^{\frac{1}{3}}=c\times\left(1+\frac{x^3}{3c^3}-\frac{2x^6}{3.6c^6}+\frac{2.5x^9}{3.6.9c^9}, \text{ \&c.}\right)$$

### EXAMPLE 8.

$$d(c^2+x^2)^{-\frac{1}{2}}.$$

Substitute $a$ for $c^2$, and $b$ for $x^2$, then

$$d(a+b)^{-\frac{1}{2}}=d\times(a^{-\frac{1}{2}}+Aa^{-\frac{3}{2}}b+Ba^{-\frac{5}{2}}b^2+Ca^{-\frac{7}{2}}b^3, \text{ \&c.})$$

$$A=-\frac{1}{2}. \qquad B=-\frac{1}{2}\times\frac{-\frac{3}{2}}{2}=-\frac{1}{2}\times-\frac{3}{4}=\frac{3}{2.4}.$$

$$C=\frac{3}{2.4}\times\frac{-\frac{5}{2}}{3}=\frac{3}{2.4}\times-\frac{5}{6}=-\frac{3.5}{2.4.6}.$$

Restoring the values of $a$ and $b$, viz. $c^2+x^2$, we have,

$$d(c^2+x^2)^{-\frac{1}{2}}=d\times\left(\frac{1}{c}-\frac{x^2}{2c^3}+\frac{3x^4}{2.4c^5}-\frac{3.5x^6}{2.4.6c^7}, \text{ \&c.}\right)$$

Or bringing $c$ out of the denominators of the series, then

$$d(c^2+x^2)^{-\frac{1}{2}}=\frac{d}{c}\times\left(1-\frac{x^2}{2c^2}+\frac{3x^4}{2.4c^4}-\frac{3.5x^6}{2.4.6c^6}, \text{ \&c.}\right)$$

### EXAMPLE 9.

$(a^2+y^3)^5$. Let $c$ represent $a^2$, and $x$ represent $y^3$:

$$(c+x)^5=c^5+5c^4x+10c^3x^2+10c^2x^3+5cx^4+x^5.$$

Restoring the values of $c$ and $x$, viz. $a^2$ and $y^3$, then

$$(x^2+y^3)^5=a^{10}+5a^8y^3+10a^6y^6+10a^4y^9+5a^2y^{12}+y^{15}.$$

### EXAMPLE 10.

$(a+b+x)^4$. Let $c$ represent $(b+x)$.

$$(a+c)^4=a^4+4a^3c+6a^2c^2+4ac^3+c^4.$$

Restoring $(b+x)$,

$$(a+b+x)^4=a^4+4a^3(b+x)+6a^2(b+x)^2+4a(b+x)^3+(b+x)^4.$$

Expanding and involving the terms which have $(b+x)$,

$$(a+b+x)^4=a^4+4a^3b+4a^3x+6a^2b^2+12a^2bx+6a^2x^2+4ab^3+ 12ab^2x+12abx^2+4ax^3+b^4+4b^3x+6b^2x^2+4bx^3+x^4.$$

### EXAMPLE 11.

$(a^3-x)^{\frac{1}{2}}$. Substitute $b$ for $a^3$, and we have

$$(b-x)^{\frac{1}{2}}=b^{\frac{1}{2}}-Ab^{-\frac{1}{2}}x+Bb^{-\frac{3}{2}}x^2-Cb^{-\frac{5}{2}}x^3+Db^{-\frac{7}{2}}x^4, \text{ \&c.}$$

The coefficients are the same as in Ex. 2 (Art. 505), for the indices are the same.

$$A=\frac{1}{2}. \quad B=-\frac{1}{2.4}. \quad C=\frac{3}{2.4.6}. \quad D=-\frac{3.5}{2.4.6.8}.$$

Restoring the value of $b$, then the series will be,

$$(a^3-x)^{\frac{1}{2}}=a^{\frac{3}{2}}-\frac{x}{2a^{\frac{3}{2}}}-\frac{x^2}{2.4a^{\frac{9}{2}}}-\frac{3x^3}{2.4.6a^{\frac{15}{2}}}-\frac{3.5x^4}{2.4.6.8a^{\frac{21}{2}}}, \text{ \&c.}$$

Or,

$$(a^3-x)^{\frac{1}{2}}-a^{\frac{3}{2}}\times\left(1-\frac{x}{2a^3}-\frac{x^2}{2.4a^6}-\frac{3x^3}{2.4.6a^9}-\frac{3.5x^4}{2.4.6.8a^{12}}, \text{ \&c.}\right)$$

### EXAMPLE 12.

$$(1-y^2)^{\frac{1}{2}}=1^{\frac{1}{2}}-A1^{-\frac{1}{2}}y^2+B1^{-\frac{3}{2}}y^4-C1^{-\frac{5}{2}}y^6+D1^{-\frac{7}{2}}y^8, \text{ \&c.}$$

$$A=\frac{1}{2}.\quad B=-\frac{1}{2.4}.\quad C=\frac{3}{2.4.6}.\quad D=-\frac{3.5}{2.4.6.8}.$$

$$(1-y^2)^{\frac{1}{2}}=1-\frac{y^2}{2}-\frac{y^4}{2.4}-\frac{3y^6}{2.4.6}-\frac{3.5y^8}{2.4.6.8}, \text{ \&c.}$$

The value of the coefficients is obtained by substituting the first power of some letter, as $a$ for $y^2$; and thus the indices being the same as those of the preceding example, the coefficients are the same.

EXAMPLE 13. $(a-x)^{\frac{1}{3}}=a^{\frac{1}{3}}\times\left(1-\frac{x}{a}\right)^{\frac{1}{3}}$.

$$a^{\frac{1}{3}}\times\left(1-\frac{x}{a}\right)^{\frac{1}{3}}=a^{\frac{1}{3}}\times\left(1^{\frac{1}{3}}-A1^{-\frac{2}{3}}\frac{x}{a}+B1^{-\frac{5}{3}}\frac{x^2}{a^2}-C1^{-\frac{8}{3}}\frac{x^3}{a^3}, \text{ \&c.}\right)$$

$$A=\frac{1}{3}.\qquad B=\frac{1}{3}\times\frac{-\frac{2}{3}}{2}=\frac{1}{3}\times-\frac{2}{6}=-\frac{2}{3.6}.$$

$$C=-\frac{2}{3.6}\times\frac{-\frac{5}{3}}{3}=-\frac{2}{3.6}\times-\frac{5}{9}=\frac{2.5}{3.6.9}.$$

$$a^{\frac{1}{3}}\times\left(1-\frac{x}{a}\right)^{\frac{1}{3}}=a^{\frac{1}{3}}\times\left(1-\frac{x}{3a}-\frac{2x^2}{3.6a^2}-\frac{2.5x^3}{3.6.9a^3}, \text{ \&c.}\right)$$

### EXAMPLE 14.

$h(a^3-y^3)^{\frac{1}{3}}$. Let $x$ stand for $a^3$, and $b$ for $y^3$.

$$h(x-b)^{\frac{1}{3}}=h\times(x^{\frac{1}{3}}-Ax^{-\frac{2}{3}}b+Bx^{-\frac{5}{3}}b^2-Cx^{-\frac{8}{3}}b^3, \text{ \&c.})$$

$$A=\frac{1}{3}.\qquad B=\frac{1}{3}\times\frac{-\frac{2}{3}}{2}=\frac{1}{3}\times-\frac{2}{6}=-\frac{2}{3.6}.$$

$$C=-\frac{2}{3.6}\times\frac{-\frac{5}{3}}{3}=-\frac{2}{3.6}\times-\frac{5}{9}=\frac{2.5}{3.6.9}.$$

Restoring the values of $x$ and $b$, we have,

$$h(a^3-y^3)^{\frac{1}{3}}=h\times\left(a-\frac{y^3}{3a^2}-\frac{2y^6}{3.6a^5}-\frac{2.5y^9}{3.6.9a^8}, \text{ \&c.}\right)$$

## EVOLUTION OF COMPOUND QUANTITIES.

### EXAMPLE 3.

Required the fifth root of

$$(a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5).$$

$$\begin{array}{rl}
 & a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5\ (a+b \\
 & a^5 \\ \cline{2-2}
5a^4) & *\quad 5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5 \\ \cline{2-2} \cline{2-2}
 & a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5 \\
 & a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5=(a+b)^5. \\ \cline{2-2}
 & *\quad *\quad *\quad *\quad *\quad *
\end{array}$$

### EXAMPLE 4.

Required the cube root of $(a^3-6a^2b+12ab^2-8b^3)$.

$$\begin{array}{rl}
 & a^3-6a^2b+12ab^2-8b^3\ (a-2b \\
 & a^3 \\ \cline{2-2}
3a^2) & *\ -6a^2b+12ab^2-8b^3 \\ \cline{2-2} \cline{2-2}
 & a^3-6a^2b+12ab^2-8b^3 \\
 & a^3-6a^2b+12ab^2-8b^3=(a-2b)^3. \\ \cline{2-2}
 & *\quad *\quad *\quad *
\end{array}$$

---

**Examples under Article 508.**

### EXAMPLE 3.

$$\begin{array}{rl}
 & a^6-2a^5+3a^4-2a^3+a^2\ (a^3-a^2+a. \\
 & a^6 \\ \cline{2-2}
2a^3-a^2) & *\ -2a^5+3a^4-2a^3+a^2 \\
 & \quad -2a^5+\ a^4 \\ \cline{2-2}
2a^3-2a^2+a) & \qquad *\ +2a^4-2a^3+a^2 \\
 & \qquad\quad +2a^3-2a^3+a^2 \\ \cline{2-2}
 & \qquad\qquad *\quad *\quad *
\end{array}$$

### EXAMPLE 4.

$$
\begin{array}{rl}
 & a^4+4a^2b+4b^2-4a^2-8b+4\ (a^2+2b-2. \\
 & \underline{a^4\qquad\qquad\qquad\qquad\qquad\qquad\qquad} \\
2a^2+2b) & *\ +4a^2b+4b^2-4a^2-8b+4 \\
 & \qquad \underline{4a^2b+4b^2\qquad\qquad\qquad\quad} \\
2a^2+4b-2) & \quad *\qquad *\ -4a^2-8b+4 \\
 & \qquad\qquad\quad \underline{-4a^2-8b+4.} \\
 & \qquad\qquad\qquad *\quad *\quad *
\end{array}
$$

---

**Promiscuous Examples.—Article 508.**

### EXAMPLE 1.

Required the square root of $(x^4-4x^3+6x^2-4x+1)$.

$$
\begin{array}{rl}
 & x^4-4x^3+6x^2-4x+1\ (x^2-2x+1. \\
 & \underline{x^4\qquad\qquad\qquad\qquad\qquad} \\
2x^2-2x) & *\ -4x^3+6x^2-4x+1 \\
 & \quad \underline{-4x^3+4x^2\qquad\qquad\quad} \\
2x^2-4x+1) & \quad *\qquad 2x^2-4x+1 \\
 & \qquad\qquad \underline{2x^2-4x+1} \\
 & \qquad\qquad\ *\quad\ *\quad\ *
\end{array}
$$

### EXAMPLE 2.

Required the cube root of $(x^6-6x^5+15x^4-20x^3+15x^2-6x+1)$.

$$
\begin{array}{rl}
 & x^6-6x^5+15x^4-20x^3+15x^2-6x+1\ (x^2-2x+1. \\
 & \underline{x^6\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad} \\
3x^4) & \underline{\underline{*\ -6x^5+15x^4-20x^3+15x^2-6x+1}} \\
 & x^6-6x^5+15x^4-20x^3+15x^2-6x+1 \\
 & \underline{x^6-6x^5+12x^4-\ 8x^3\qquad\qquad\qquad\qquad =(x^2-2x)^3.} \\
3x^4) & \underline{\underline{*\qquad *\qquad 3x^4-12x^3+15x^2-6x+1}} \\
 & x^6-6x^5+15x^4-20x^3+15x^2-6x+1 \\
 & \underline{x^6-6x^5+15x^4-20x^3+15x^2-6x+1=(x^2-2x+1)^3.} \\
 & *\quad\ *\quad\ *\quad\ *\quad\ *\quad\ *\quad\ *
\end{array}
$$

**Ex. 3.** Required the square root of $\sqrt{(4x^4-4x^3+13x^2-6x+9)}$.

$$\begin{array}{rl}
 & 4x^4-4x^3+13x^2-6x+9\,(2x^2-x+3. \\
 & 4x^4 \\ \hline
4x^2-x) & *\ -4x^3+13x^2-6x+9 \\
 & \quad -4x^3+\ \ x^2 \\ \hline
4x^2-2x+3) & \quad *\ +12x^2-6x+9 \\
 & \quad\quad +12x^2-6x+9 \\ \hline
 & \quad\quad\quad *\quad *\quad *
\end{array}$$

**Ex. 4.** Required the fourth root of
$(16a^4-96a^3x+216a^2x^2-216ax^3+81x^4)$.

$$\begin{array}{rl}
 & 16a^4-96a^3x+216a^2x^2-216ax^3+81x^4\,(2a-3x. \\
 & 16a^4 \\ \hline
32a^3) & *\ \ -96a^3x+216a^2x^2-216ax^3+81x^4 \\ \hline\hline
 & 16a^4-96a^3x+216a^2x^2-216ax^3+81x^4 \\
 & 16a^4-96a^3x+216a^2x^2-216ax^3+81x^4=(2a-3x)^4. \\ \hline
 & *\quad *\quad *\quad *\quad *
\end{array}$$

**Ex. 5.** Required the 5th root of $(x^5+5x^4+10x^3+10x^2+5x+1)$.

$$\begin{array}{rl}
 & x^5+5x^4+10x^3+10x^2+5x+1\,(x+1. \\
 & x^5 \\ \hline
5x^4) & *\ \ 5x^4+10x^3+10x^2+5x+1 \\ \hline\hline
 & x^5+5x^4+10x^3+10x^2+5x+1 \\
 & x^5+5x^4+10x^3+10x^2+5x+1=(x+1)^5. \\ \hline
 & *\quad *\quad *\quad *\quad *\quad *.
\end{array}$$

**Ex. 6.** Required the 6th root of
$(a^6-6a^5b+15a^4b^2-20a^3b^3+15a^2b^4-6ab^5+b^6)$.

$$\begin{array}{rl}
 & a^6-6a^5b+15a^4b^2-20a^3b^3+15a^2b^4-6ab^5+b^6\,(a-b. \\
 & a^6 \\ \hline
6a^5) & *\ -6a^5b+15a^4b^2-20a^3b^3+15a^2b^4-6ab^5+b^6 \\ \hline\hline
 & a^6-6a^5b+15a^4b^2-20a^3b^3+15a^2b^4-6ab^5+b^6 \\
 & a^6-6a^5b+15a^4b^2-20a^3b^3+15a^2b^4-6ab^5+b^6=(a-b)^6. \\ \hline
 & *\quad *\quad *\quad *\quad *\quad *
\end{array}$$

## ROOTS OF BINOMIAL SURDS.

**Ex. 1.** Find the square root of $3+2\sqrt{2}$.

$$a=3. \qquad \sqrt{b}=2\sqrt{2}.$$

Then, $a^2=9$. $b=(2\sqrt{2})^2=8$. $a^2-b=9-8=1$.

By formula 1st, $\sqrt{3+2\sqrt{2}}=\sqrt{\frac{3+1}{2}}+\sqrt{\frac{3-1}{2}}$.

$$\sqrt{\frac{3+1}{2}}=\sqrt{\frac{4}{2}}=\sqrt{2}. \qquad \sqrt{\frac{3-1}{2}}=\sqrt{\frac{2}{2}}=\sqrt{1}=1.$$

Then, $\sqrt{3+2\sqrt{2}}=\sqrt{2}+1$

**Ex. 2.** Find the square root of $11+6\sqrt{2}$.

$$a=11. \qquad \sqrt{b}=6\sqrt{2}.$$

Then, $a^2=121$. $b=(6\sqrt{2})^2=72$. $a^2-b=121-72=49$.

$$\sqrt{a^2-b}=\sqrt{49}=7.$$

By formula 1st, we have, $\sqrt{11+6\sqrt{2}}=\sqrt{\frac{11+7}{2}}+\sqrt{\frac{11-7}{2}}$.

$$\sqrt{\frac{11+7}{2}}=\sqrt{\frac{18}{2}}=\sqrt{9}=3. \qquad \sqrt{\frac{11-7}{2}}=\sqrt{\frac{4}{2}}=\sqrt{2}.$$

Then, $\sqrt{11+6\sqrt{2}}=3+\sqrt{2}$.

**Ex. 3.** Find the square root of $6-2\sqrt{5}$.

$a=6$. $\sqrt{b}=2\sqrt{5}$. $a^2=36$. $b=(2\sqrt{5})^2=20$.

$a^2-b=36-20=16$. $\sqrt{a^2-b}=\sqrt{16}=4$.

By formula 2d, $\sqrt{6-2\sqrt{5}}=\sqrt{\frac{6+4}{2}}-\sqrt{\frac{6-4}{2}}$.

$$\sqrt{\frac{6+4}{2}}=\sqrt{\frac{10}{2}}=\sqrt{5}. \qquad \sqrt{\frac{6-4}{2}}=\sqrt{\frac{2}{2}}=\sqrt{1}=1.$$

Then, $\sqrt{6-2\sqrt{5}}=\sqrt{5}-1$.

Ex. 4. Find the square root of $7+4\sqrt{3}$.

$a=7.$ $\sqrt{b}=4\sqrt{3}.$ $a^2=49.$

$b=(4\sqrt{3})^2=48.$ $a^2-b=49-48=1.$

By formula 1st, $\sqrt{7+4\sqrt{3}}=\sqrt{\frac{7+1}{2}}+\sqrt{\frac{7-1}{2}}.$

$\sqrt{\frac{7+1}{2}}=\sqrt{\frac{8}{2}}=\sqrt{4}=2.$ $\sqrt{\frac{7-1}{2}}=\sqrt{\frac{6}{2}}=\sqrt{3}.$

Then, $\sqrt{7+4\sqrt{3}}=2+\sqrt{3}.$

Ex. 5. Find the square root of $7-2\sqrt{10}$.

$a=7.$ $\sqrt{b}=2\sqrt{10}.$ $a^2=49.$ $b=(2\sqrt{10})^2=40.$

$a^2-b=49-40=9.$ $\sqrt{a^2-b}=\sqrt{9}=3.$

By formula 2d, $\sqrt{7-2\sqrt{10}}=\sqrt{\frac{7+3}{2}}-\sqrt{\frac{7-3}{2}}.$

$\sqrt{\frac{7+3}{2}}=\sqrt{\frac{10}{2}}=\sqrt{5}.$ $\sqrt{\frac{7-3}{2}}=\sqrt{\frac{4}{2}}=\sqrt{2}.$

Then, $\sqrt{7-2\sqrt{10}}=\sqrt{5}-\sqrt{2}.$

◆

## INFINITE SERIES.

**Art. 512.** Ex. 4. Reduce $\frac{1+a}{1-a}$ to an infinite series.

$$\begin{array}{l}
1-a)\,1+a\,(1+2a+2a^2+2a^3+2a^4+2a^5, \text{\&c.}\\
\qquad\underline{1-a}\\
\qquad\quad 2a\\
\qquad\quad \underline{2a-2a^2}\\
\qquad\qquad\quad 2a^2\\
\qquad\qquad\quad \underline{2a^2-2a^3}\\
\qquad\qquad\qquad\quad 2a^3\\
\qquad\qquad\qquad\quad \underline{2a^3-2a^4}\\
\qquad\qquad\qquad\qquad\quad 2a^4\\
\qquad\qquad\qquad\qquad\quad \underline{2a^4-2a^5}\\
\qquad\qquad\qquad\qquad\qquad\quad 2a^5
\end{array}$$

It is evident that if the operation be continued in the same manner the series will not terminate, and each term, after the second, will be equal to the preceding term multiplied by $a$.

**Art. 513.** Ex. 2. Reduce $\sqrt{a^2-b^2}$ to an infinite series.

$$\begin{array}{r l}
 & a^2-b^2\left(a-\dfrac{b^2}{2a}-\dfrac{b^4}{8a^3}-\dfrac{b^6}{16a^5},\ \&c.\right. \\
 & \underline{a^2} \\
2a-\dfrac{b^2}{2a}\Bigg) & *\ -b^2 \\
 & \underline{-b^2+\dfrac{b^4}{4a^2}} \\
2a-\dfrac{b^2}{a}-\dfrac{b^4}{8a^3}\Bigg) & *\ -\dfrac{b^4}{4a^2} \\
 & \underline{-\dfrac{b^4}{4a^2}+\dfrac{b^6}{8a^4}+\dfrac{b^8}{64a^6}} \\
2a-\dfrac{b^2}{a}-\dfrac{b^4}{4a^3}-\dfrac{b^6}{16a^5}\Bigg) & *\ -\dfrac{b^6}{8a^4}-\dfrac{b^8}{64a^6},\ \&c.
\end{array}$$

Ex. 3. Reduce $\sqrt{2}$, or $\sqrt{1+1}$, to an infinite series.

$$\begin{array}{r l}
 & 1+1\left(1+\dfrac{1}{2}-\dfrac{1}{8}+\dfrac{1}{16},\ \&c.\right. \\
 & \underline{1} \\
2+\dfrac{1}{2}\Bigg) & *+1 \\
 & \underline{+1+\dfrac{1}{4}} \\
2+1-\dfrac{1}{8}\Bigg) & *-\dfrac{1}{4} \\
 & \underline{-\dfrac{1}{4}-\dfrac{1}{8}+\dfrac{1}{64}} \\
2+1-\dfrac{1}{4}+\dfrac{1}{16}\Bigg) & *+\dfrac{1}{8}-\dfrac{1}{64},\ \&c.
\end{array}$$

**Ex. 4.** Reduce $\sqrt{1+x}$ to an infinite series.

$$\begin{array}{rl}
 & 1+x\left(1+\frac{x}{2}-\frac{x^2}{8}+\frac{x^3}{16}-\frac{5x^4}{128},\ \text{\&c.}\right. \\
 & \underline{1} \\
2+\frac{x}{2}\Big) & *+x \\
 & \underline{+x+\frac{x^2}{4}} \\
2+x-\frac{x^2}{8}\Big) & *-\frac{x^2}{4} \\
 & \underline{-\frac{x^2}{4}-\frac{x^3}{8}+\frac{x^4}{64}} \\
2+x-\frac{x^2}{4}+\frac{x^3}{16}\Big) & *+\frac{x^3}{8}-\frac{x^4}{64} \\
 & \underline{+\frac{x^3}{8}+\frac{x^4}{16}-\frac{x^5}{64}+\frac{x^6}{256}} \\
2+x-\frac{x^2}{4}+\frac{x^3}{8}-\frac{5x^4}{128}\Big) & -\frac{5x^4}{64}+\frac{x^5}{64}-\frac{x^6}{256},\ \text{\&c.}
\end{array}$$

---

## Indeterminate Coefficients.

**Ex. 3.** Expand into a series $\frac{1+2x}{1-x-x^2}$.

Assume $\frac{1+2x}{1-x-x^2}=A+Bx+Cx^2+Dx^3+Ex^4$, &c.

Clear. of fractions, $1+2x=A-Ax-Ax^2+Bx-Bx^2-Bx^3+Cx^2-Cx^3-Cx^4+Dx^3-Dx^4-Dx^5+Ex^4$, &c.

Transposing $1+2x$, and collecting the several powers of $x$,

$$0=(A-1)+(B-A-2)x+(C-A-B)x^2+(D-C-B)x^3+(E-D-C)x^4,\ \text{\&c.}$$

Equating the coefficients of the different powers of $x$ with zero, and finding the values of $A$, $B$, $C$, $D$, &c.

### EXAMPLE 12.

$$(1-y^2)^{\frac{1}{2}}=1^{\frac{1}{2}}-A1^{-\frac{1}{2}}y^2+B1^{-\frac{3}{2}}y^4-C1^{-\frac{5}{2}}y^6+D1^{-\frac{7}{2}}y^8, \text{ \&c.}$$

$$A=\frac{1}{2}. \qquad B=-\frac{1}{2.4}. \qquad C=\frac{3}{2.4.6}. \qquad D=-\frac{3.5}{2.4.6.8}.$$

$$(1-y^2)^{\frac{1}{2}}=1-\frac{y^2}{2}-\frac{y^4}{2.4}-\frac{3y^6}{2.4.6}-\frac{3.5y^8}{2.4.6.8}, \text{ \&c.}$$

The value of the coefficients is obtained by substituting the first power of some letter, as $a$ for $y^2$; and thus the indices being the same as those of the preceding example, the coefficients are the same.

EXAMPLE 13. $(a-x)^{\frac{1}{3}}=a^{\frac{1}{3}}\times\left(1-\frac{x}{a}\right)^{\frac{1}{3}}.$

$$a^{\frac{1}{3}}\times\left(1-\frac{x}{a}\right)^{\frac{1}{3}}=a^{\frac{1}{3}}\times\left(1^{\frac{1}{3}}-A1^{-\frac{2}{3}}\frac{x}{a}+B1^{-\frac{5}{3}}\frac{x^2}{a^2}-C1^{-\frac{8}{3}}\frac{x^3}{a^3}, \text{ \&c.}\right)$$

$$A=\frac{1}{3}. \qquad B=\frac{1}{3}\times\frac{-\frac{2}{3}}{2}=\frac{1}{3}\times-\frac{2}{6}=-\frac{2}{3.6}.$$

$$C=-\frac{2}{3.6}\times\frac{-\frac{5}{3}}{3}=-\frac{2}{3.6}\times-\frac{5}{9}=\frac{2.5}{3.6.9}.$$

$$a^{\frac{1}{3}}\times\left(1-\frac{x}{a}\right)^{\frac{1}{3}}=a^{\frac{1}{3}}\times\left(1-\frac{x}{3a}-\frac{2x^2}{3.6a^2}-\frac{2.5x^3}{3.6.9a^3}, \text{ \&c.}\right)$$

### EXAMPLE 14.

$h(a^3-y^2)^{\frac{1}{3}}$. Let $x$ stand for $a^3$, and $b$ for $y^2$.

$$h(x-b)^{\frac{1}{3}}=h\times(x^{\frac{1}{3}}-Ax^{-\frac{2}{3}}b+Bx^{-\frac{5}{3}}b^2-Cx^{-\frac{8}{3}}b^3, \text{ \&c.})$$

$$A=\frac{1}{3}. \qquad B=\frac{1}{3}\times\frac{-\frac{2}{3}}{2}=\frac{1}{3}\times-\frac{2}{6}=-\frac{2}{3.6}.$$

$$C=-\frac{2}{3.6}\times\frac{-\frac{5}{3}}{3}=-\frac{2}{3.6}\times-\frac{5}{9}=\frac{2.5}{3.6.9}.$$

Restoring the values of $x$ and $b$, we have,

$$h(a^3-y^3)^{\frac{1}{3}}=h\times\left(a-\frac{y^3}{3a^2}-\frac{2y^6}{3.6a^5}-\frac{2.5y^9}{3.6.9a^8}, \text{ \&c.}\right)$$

## EVOLUTION OF COMPOUND QUANTITIES.

### EXAMPLE 3.

Required the fifth root of

$$(a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5).$$

$$\begin{array}{rl}
 & a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5\ (a+b \\
 & a^5 \\ \cline{2-2}
5a^4) & *\quad 5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5 \\ \cline{2-2} \cline{2-2}
 & a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5 \\
 & a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5=(a+b)^5. \\ \cline{2-2}
 & *\quad\ *\quad\ *\quad\ *\quad\ *\quad\ *
\end{array}$$

### EXAMPLE 4.

Required the cube root of $(a^3-6a^2b+12ab^2-8b^3)$.

$$\begin{array}{rl}
 & a^3-6a^2b+12ab^2-8b^3\ (a-2b \\
 & a^3 \\ \cline{2-2}
3a^2) & *\ -6a^2b+12ab^2-8b^3 \\ \cline{2-2} \cline{2-2}
 & a^3-6a^2b+12ab^2-8b^3 \\
 & a^3-6a^2b+12ab^2-8b^3=(a-2b)^3. \\ \cline{2-2}
 & *\quad\ *\quad\ *\quad\ *
\end{array}$$

---

**Examples under Article 508.**

### EXAMPLE 3.

$$\begin{array}{rl}
 & a^6-2a^5+3a^4-2a^3+a^2\ (a^3-a^2+a. \\
 & a^6 \\ \cline{2-2}
2a^3-a^2) & *\ -2a^5+3a^4-2a^3+a^2 \\
 & \quad\ -2a^5+\ \ a^4 \\ \cline{2-2}
2a^3-2a^2+a) & \qquad\ *\ +2a^4-2a^3+a^2 \\
 & \qquad\quad\ +2a^3-2a^3+a^2 \\ \cline{2-2}
 & \qquad\qquad *\quad\ *\quad\ *
\end{array}$$

### EXAMPLE 4.

$$\begin{array}{rl}
 & a^4+4a^2b+4b^2-4a^2-8b+4\,(a^2+2b-2. \\
 & a^4 \\ \cline{2-2}
2a^2+2b) & *\;+4a^2b+4b^2-4a^2-8b+4 \\
 & \quad 4a^2b+4b^2 \\ \cline{2-2}
2a^2+4b-2) & *\qquad *\;-4a^2-8b+4 \\
 & \qquad\qquad -4a^2-8b+4. \\ \cline{2-2}
 & \qquad\qquad\; * \quad\; * \quad\; *
\end{array}$$

---

**Promiscuous Examples.—Article 508.**

### EXAMPLE 1.

Required the square root of $(x^4-4x^3+6x^2-4x+1)$.

$$\begin{array}{rl}
 & x^4-4x^3+6x^2-4x+1\,(x^2-2x+1. \\
 & x^4 \\ \cline{2-2}
2x^2-2x) & *\;-4x^3+6x^2-4x+1 \\
 & \quad -4x^3+4x^2 \\ \cline{2-2}
2x^2-4x+1) & \quad *\quad 2x^2-4x+1 \\
 & \qquad\quad 2x^2-4x+1 \\ \cline{2-2}
 & \qquad\quad * \quad\; * \quad\; *
\end{array}$$

### EXAMPLE 2.

Required the cube root of $(x^6-6x^5+15x^4-20x^3+15x^2-6x+1)$.

$$\begin{array}{rl}
 & x^6-6x^5+15x^4-20x^3+15x^2-6x+1\,(x^2-2x+1. \\
 & x^6 \\ \cline{2-2}
3x^4) & *\;-6x^5+15x^4-20x^3+15x^2-6x+1 \\ \hline\hline
 & x^6-6x^5+15x^4-20x^3+15x^2-6x+1 \\
 & x^6-6x^5+12x^4-\;8x^3 \qquad\qquad\quad =(x^2-2x)^3. \\ \cline{2-2}
3x^4) & *\quad *\quad 3x^4-12x^3+15x^2-6x+1 \\ \hline\hline
 & x^6-6x^5+15x^4-20x^3+15x^2-6x+1 \\
 & x^6-6x^5+15x^4-20x^3+15x^2-6x+1=(x^2-2x+1)^3. \\ \cline{2-2}
 & * \quad\; * \quad\; * \quad\; * \quad\; * \quad\; * \quad\; *
\end{array}$$

**Ex. 3.** Required the square root of $(4x^4-4x^3+13x^2-6x+9)$.

$$\begin{array}{rl}
 & 4x^4-4x^3+13x^2-6x+9\ (2x^2-x+3. \\
 & \underline{4x^4\phantom{-4x^3+13x^2-6x+9}} \\
4x^2-x) & *\ -4x^3+13x^2-6x+9 \\
 & \underline{\phantom{*}\ -4x^3+\phantom{1}x^2\phantom{-6x+9}} \\
4x^2-2x+3) & \phantom{*-4x^3}*\ +12x^2-6x+9 \\
 & \underline{\phantom{*-4x^3*}\ +12x^2-6x+9} \\
 & \phantom{*-4x^3*+12}*\quad *\quad *
\end{array}$$

**Ex. 4.** Required the fourth root of

$$(16a^4-96a^3x+216a^2x^2-216ax^3+81x^4).$$

$$\begin{array}{rl}
 & 16a^4-96a^3x+216a^2x^2-216ax^3+81x^4\ (2a-3x. \\
 & \underline{16a^4\phantom{-96a^3x+216a^2x^2-216ax^3+81x^4}} \\
32a^3) & \underline{\underline{*\quad -96a^3x+216a^2x^2-216ax^3+81x^4}} \\
 & 16a^4-96a^3x+216a^2x^2-216ax^3+81x^4 \\
 & \underline{16a^4-96a^3x+216a^2x^2-216ax^3+81x^4=(2a-3x)^4.} \\
 & *\quad *\quad *\quad *\quad *
\end{array}$$

**Ex. 5.** Required the 5th root of $(x^5+5x^4+10x^3+10x^2+5x+1)$.

$$\begin{array}{rl}
 & x^5+5x^4+10x^3+10x^2+5x+1\ (x+1. \\
 & \underline{x^5\phantom{+5x^4+10x^3+10x^2+5x+1}} \\
5x^4) & \underline{\underline{*\quad 5x^4+10x^3+10x^2+5x+1}} \\
 & x^5+5x^4+10x^3+10x^2+5x+1 \\
 & \underline{x^5+5x^4+10x^3+10x^2+5x+1=(x+1)^5.} \\
 & *\quad *\quad *\quad *\quad *\quad *
\end{array}$$

**Ex. 6.** Required the 6th root of

$$(a^6-6a^5b+15a^4b^2-20a^3b^3+15a^2b^4-6ab^5+b^6).$$

$$\begin{array}{rl}
 & a^6-6a^5b+15a^4b^2-20a^3b^3+15a^2b^4-6ab^5+b^6\ (a-b. \\
 & \underline{a^6\phantom{-6a^5b+15a^4b^2-20a^3b^3+15a^2b^4-6ab^5+b^6}} \\
6a^5) & \underline{\underline{*\ -6a^5b+15a^4b^2-20a^3b^3+15a^2b^4-6ab^5+b^6}} \\
 & a^6-6a^5b+15a^4b^2-20a^3b^3+15a^2b^4-6ab^5+b^6 \\
 & \underline{a^6-6a^5b+15a^4b^2-20a^3b^3+15a^2b^4-6ab^5+b^6=(a-b)^6.} \\
 & *\quad *\quad *\quad *\quad *\quad *
\end{array}$$

And reducing,

$$\begin{aligned}
&A=a\\
&B=2Ad+b=2ad+b\\
&C=2Bd-Ad^2=2bd+4ad^2-ad^2=2bd+3ad^2\\
&D=2Cd-Bd^2=4bd^2+6ad^3-2ad^3-bd^2=3bd^2+4ad^3.
\end{aligned}$$

It is evident that if to the terms of the coefficient of any term of the series, except the first, there be added respectively $a$ and $b$, and the sum multiplied by $d$, the result will be the coefficient of the following term.

Subst. the values of $A$, $B$, $C$, &c., in the assumed series,

$$\frac{a+bx}{(1-dx)^2}=a+(b+2ad)x+(2bd+3ad^2)x^2+(3bd^2+4ad^3)x^3 \\ +(4bd^3+5ad^4)x^4, \text{ \&c.}$$

Ex. 10. Expand into a series $\frac{1+x}{(1-x)^3}=\frac{1+x}{1-3x+3x^2-x^3}$.

Assume $\frac{1+x}{1-3x+3x^2-x^3}=A+Bx+Cx^2+Dx^3+Ex^4$, &c.

Clear. of fract., $1+x=A-3Ax+3Ax^2-Ax^3+Bx-3Bx^2+3Bx^3-Bx^4+Cx^2-3Cx^3+3Cx^4-Cx^5+Dx^3$, &c.

Transposing $(1+x)$, and collecting the several powers of $x$,

$$0=(A-1)+(B-1-3A)x+(C-3B+3A)x^2+ \\ (D-3C+3B-A)x^3+(E-3D+3C-B)x^4, \text{ \&c.}$$

Equating the coefficients of the different powers of $x$ with zero, and reducing,

$$\begin{array}{rl}
A-1=0 & A=1\\
B-1-3A=0 & B=1+3A=4\\
C-3B+3A=0 & C=3B-3A=12-3=9\\
D-3C+3B-A=0 & D=3C-3B+A=27-12+1=16\\
E-3D+3C-B=0 & E=3D-3C+B=48-27+4=25.
\end{array}$$

Subst. these values of $A$, $B$, $C$, &c., in the assumed series,

$$\frac{1+x}{(1-x)^3}=1+4x+9x^2+16x^3+25x^4+36x^5, \text{ \&c.}$$

**In this series it is evident that the coefficients are the squares of the natural numbers in their regular order.**

The Examples under Art. 517 are so simple, that it is unnecessary to present solutions of them.

## Recurring Series.

### EXAMPLE 1.

Find the sum of the series, $1+6x+12x^2+48x^3+120x^4$, &c.

$$\left\{\begin{matrix} A & B & C & D & E & \\ 1 & +6x & +12x^2 & +48x^3 & +120x^4, & \&c. \end{matrix}\right\}$$

Substituting in the formulas of Art. 519,

$$m=\frac{DC-BE}{CC-BD}=\frac{48\times12-6\times120}{12^2-6\times48}=\frac{576-720}{144-288}=\frac{-144}{-144}=1.$$

$$n=\frac{CE-DD}{CC-BD}=\frac{12\times120-48^2}{12^2-6\times48}=\frac{1440-2304}{144-288}=\frac{-864}{-144}=6.$$

$m=1$. $n=6$. $A=1$. $B=6x$. Scale of relation $=1+6$.

$$S=\frac{A+B-Amx}{1-mx-nx^2}=\frac{1+6x-x}{1-x-6x^2}=\frac{1+5x}{1-x-6x^2}.$$

### EXAMPLE 2.

Find the sum of the series, $1+3x+4x^2+7x^3+11x^4$, &c.

$$\left\{\begin{matrix} A & B & C & D & E & \\ 1 & +3x & +4x^2 & +7x^3 & +11x^4, & \&c. \end{matrix}\right\}$$

Substituting in the formulas,

$$m=\frac{DC-BE}{CC-BD}=\frac{7\times4-3\times11}{4^2-3\times7}=\frac{28-33}{16-21}=\frac{-5}{-5}=1.$$

$$n=\frac{CE-DD}{CC-BD}=\frac{4\times11-7^2}{4^2-3\times7}=\frac{44-49}{16-21}=\frac{-5}{-5}=1.$$

$m=1$. $n=1$. $A=1$. $B=3x$. Scale of relation $=1+1$.

$$S=\frac{A+B-Amx}{1-mx-nx^2}=\frac{1+3x-x}{1-x-x^2}=\frac{1+2x}{1-x-x^2}.$$

### EXAMPLE 3.

Find the sum of the series, $1+x+5x^2+13x^3+41x^4$, &c.

$$\left\{\begin{array}{l} A \quad B \quad\; C \qquad D \qquad\; E \\ 1+x+5x^2+13x^3+41x^4,\text{ \&c.}\end{array}\right\}$$

Substituting in the formulas,

$$m=\frac{DC-BE}{CC-BD}=\frac{13\times5-41}{5^2-13}=\frac{65-41}{25-13}=\frac{24}{12}=2.$$

$$n=\frac{CE-DD}{CC-BD}=\frac{5\times41-13^2}{5^2-13}=\frac{205-169}{25-13}=\frac{36}{12}=3.$$

$m=2.$ $n=3.$ $A=1.$ $B=x.$ Scale of relation $=2+3.$

$$S=\frac{A+B-Amx}{1-mx-nx^2}=\frac{1+x-2x}{1-2x-3x^2}=\frac{1-x}{1-2x-3x^2}.$$

### EXAMPLE 4.

Find the sum of the series, $1+2x+3x^2+4x^3+5x^4$, &c.

$$\left\{\begin{array}{l} A \quad\; B \qquad C \qquad D \qquad E \\ 1+2x+3x^2+4x^3+5x^4,\text{ \&c.}\end{array}\right\}$$

Substituting in the formulas,

$$m=\frac{DC-BE}{CC-BD}=\frac{4\times3-2\times5}{3^2-4\times2}=\frac{12-10}{9-8}=\frac{2}{1}=2.$$

$$n=\frac{CE-DD}{CC-BD}=\frac{3\times5-4^2}{3^2-2\times4}=\frac{15-16}{9-8}=\frac{-1}{1}=-1.$$

$m=2.$ $n=-1.$ $A=1.$ $B=2x.$ Scale of relation $=2-1.$

$$S=\frac{A+B-Amx}{1-mx-nx^2}=\frac{1+2x-2x}{1-2x+x^2}=\frac{1}{1-2x+x^2}=\frac{1}{(1-x)^2}.$$

---

## Method of Differences.

### EXAMPLE 1.

Find the *n*th term of the series, 1, 3, 6, 10, 15, 21, &c.

| | |
|---|---|
| Proposed series, | 1, 3, 6, 10, 15, 21, &c. |
| First order of diff. | 2, 3, 4, 5, 6, &c. |
| Second do. | 1, 1, 1, 1, &c. |
| Third do. | 0, 0, 0. |

$a=1.$ $D'=2.$ $D''=1.$ $D'''=0$

Substituting in the formula, Art. 522,

$$\text{The } n\text{th term } =1+(n-1)2+(n-1)\frac{n-2}{2}\times 1$$

$$=1+2n-2+\frac{n^2-3n+2}{2}$$

$$=\tfrac{1}{2}(2+4n-4+n^2-3n+2)=\tfrac{1}{2}(n^2+n).$$

The 20*th* term $=\frac{1}{2}(20^2+20)=210.$

The 50*th* term $=\frac{1}{2}(50^2+50)=1275.$

### EXAMPLE 2.

Find the 20*th* term of the series, $1^3, 2^3, 3^3, 4^3, 5^3$, &c.

| | |
|---|---|
| Proposed series, | 1, 8, 27, 64, 125, &c. |
| First order of diff. | 7, 19, 37, 61, &c. |
| Second do. | 12, 18, 24, &c. |
| Third do. | 6, 6, &c. |
| Fourth do. | 0. |

$a=1.$ $D'=7.$ $D''=12.$ $D'''=6.$ $D''''=0.$ $n=20.$

$$\text{The 20th term } =1+(20-1)7+(20-1)\frac{20-2}{2}\times 12+(20-1)\frac{20-2}{2}\times\frac{20-3}{3}\times 6=1+19\times 7+19\times 9\times 12+19\times 9\times \tfrac{17}{3}\times 6$$

$$=1+133+2052+5814=8000.$$

### EXAMPLE 3.

Find the 12*th* term of the series, 2, 6, 12, 20, 30, &c.

| | |
|---|---|
| Proposed series, | 2, 6, 12, 20, 30, &c. |
| First order of diff. | 4, 6, 8, 10, &c. |
| Second do. | 2, 2, 2, &c. |
| Third do. | 0, 0. |

$a=2.$ $D'=4.$ $D''=2.$ $D'''=0.$ $n=12.$

$$\text{The 12th term } =2+(12-1)4+(12-1)\frac{12-2}{2}\times 2$$

$$=2+11\times 4+11\times 5\times 2=2+44+110=156.$$

Ex. 4. Find the 15*th* term of the series, $1^2$, $2^2$, $3^2$, $4^2$, $5^2$, &c.

| | |
|---|---|
| Proposed series, | 1, 4, 9, 16, 25, &c. |
| First order of diff. | 3, 5, 7, 9, &c. |
| Second do. | 2, 2, 2, &c. |
| Third do. | 0, 0. |

$a=1.\quad D'=3.\quad D''=2.\quad D'''=0.\quad n=15.$

The 15*th* term $=1+(15-1)3+(15-1)\frac{15-2}{2}\times 2$

$$=1+14\times 3+14\times \tfrac{13}{2}\times 2=1+42+182=225.$$

### ARTICLE 523. EXAMPLE 2.

Find the sum of $n$ terms of the series, $1^2$, $2^2$, $3^2$, $4^2$, $5^2$, &c.

| | |
|---|---|
| Proposed series, | 1, 4, 9, 16, 25, &c. |
| First order of diff. | 3, 5, 7, 9, &c. |
| Second do. | 2, 2, 2, &c. |
| Third do. | 0, 0. |

$a=1.\qquad D'=3.\qquad D''=2.\qquad D'''=0.$

$$S=n+n\left(\frac{n-1}{2}\right)\times 3+n\left(\frac{n-1}{2}\right)\times\left(\frac{n-2}{3}\right)\times 2.$$

Reducing, $S=n+\frac{3}{2}(n^2-n)+\frac{1}{3}(n^3-3n^2+2n)$.

Clearing of fractions,

$$6S=6n+9n^2-9n+2n^3-6n^2+4n=2n^3+3n^2+n.$$

Dividing by 6, then $S=\frac{1}{6}(2n^3+3n^2+n)$.

If $n=20$, then $S=\frac{1}{6}(2\times 20^3+3\times 20^2+20)$

$$=\tfrac{1}{6}(16000+1200+20)=2870.$$

### EXAMPLE 3.

Find the sum of $n$ terms of the series, $1^3$, $2^3$, $3^3$, $4^3$ $5^3$, &c.

| | |
|---|---|
| Proposed series, | 1, 8, 27, 64, 125, &c. |
| First order of diff. | 7, 19, 37, 61, &c. |
| Second do. | 12, 18, 24, &c. |
| Third do. | 6, 6, &c. |
| Fourth do. | 0. |

$a=1.\quad D'=7.\quad D''=12.\quad D'''=6.\quad D''''=0.$

Then, $S=n+n\left(\frac{n-1}{2}\right)\times 7+n\left(\frac{n-1}{2}\right)\times\left(\frac{n-2}{3}\right)\times 12+n\left(\frac{n-1}{2}\right)\times\left(\frac{n-2}{3}\right)\times\left(\frac{n-3}{4}\right)\times 6.$

Reducing, $S=$

$$n+\tfrac{7}{2}(n^2-n)+(n^3-3n^2+2n)\times 2+(n^4-6n^3+11n^2-6n)\times\tfrac{1}{4}.$$

$$4S=4n+14n^2-14n+8n^3-24n^2+16n+n^4-6n^3+11n^2-6n.$$

Reducing, $4S=n^4+2n^3+n^2$; and $S=\frac{1}{4}(n^4+2n^3+n^2)$.

If $n$ is equal to 50, then $S=\frac{1}{4}(50^4+2\times 50^3+50^2)$.

Expanding, $S=\frac{1}{4}(6250000+250000+2500)=1625625.$

### EXAMPLE 4.

Find the sum of $n$ terms of the series, 2, 6, 12, 20, 30, &c.

| | |
|---|---|
| Proposed series, | 2, 6, 12, 20, 30, &c. |
| First order of diff. | 4, 6, 8, 10, &c. |
| Second do. | 2, 2, 2, &c. |
| Third do. | 0, 0. |

$a=2.$ $D'=4.$ $D''=2.$ $D'''=0.$

$$S=2n+n\left(\frac{n-1}{2}\right)\times 4+n\left(\frac{n-1}{2}\right)\times\left(\frac{n-2}{3}\right)\times 2.$$

Reducing, $S=2n+(n^2-n)\times 2+\frac{1}{3}(n^3-3n^2+2n)$.

Clearing of fractions, $3S=6n+6n^2-6n+n^3-3n^2+2n$

$$=n^3+3n^2+2n=n(n^2+3n+2).$$

But $(n^2+3n+2)=(n+1)\times(n+2)$.

Then, $3S=n(n+1)\times(n+2)$.

Dividing by 3, then, $S=\frac{1}{3}n(n+1)\times(n+2)$.

### EXAMPLE 5.

Find the sum of 20 terms of the series, 1, 3, 6, 10, 15, &c.

| | |
|---|---|
| Proposed series, | 1, 3, 6, 10, 15, &c. |
| First order of diff. | 2, 3, 4, 5, &c. |
| Second do. | 1, 1, 1, &c. |
| Third do. | 0, 0. |

$a=1. \quad D'=2. \quad D''=1. \quad D'''=0. \quad n=20.$

$$S=20+20\left(\frac{20-1}{2}\right)\times 2+20\left(\frac{20-1}{2}\right)\times\left(\frac{20-2}{3}\right)\times 1.$$

Reducing,

$$S=20+20\times 19+20\times 19\times 18\times\tfrac{1}{6}=20+380+1140=1540.$$

### EXAMPLE 6.

Find the sum of 12 terms of the series, $1^4, 2^4, 3^4, 4^4, 5^4, 6^4$, &c.

| | |
|---|---|
| Proposed series, | 1, 16, 81, 256, 625, 1296, &c. |
| First order of diff. | 15, 65, 175, 369, 671, &c. |
| Second do. | 50, 110, 194, 302, &c. |
| Third do. | 60, 84, 108, &c. |
| Fourth do. | 24, 24, &c. |
| Fifth do. | 0. |

$a=1. \quad D'=15. \quad D''=50. \quad D'''=60.$
$D''''=24. \quad D'''''=0. \quad n=12.$

$$S=12+12\left(\frac{12-1}{2}\right)\times 15+12\left(\frac{12-1}{2}\right)\times\left(\frac{12-2}{3}\right)\times 50+ 12\left(\frac{12-1}{2}\right)\times\left(\frac{12-2}{3}\right)\times\left(\frac{12-3}{4}\right)\times 60+12\left(\frac{12-1}{2}\right)\times \left(\frac{12-2}{3}\right)\times\left(\frac{12-3}{4}\right)\times\left(\frac{12-4}{5}\right)\times 24.$$

$$S=12+12\times\tfrac{11}{2}\times 15+12\times\tfrac{11}{2}\times\tfrac{10}{3}\times 50+12\times\tfrac{11}{2}\times\tfrac{10}{3}\times\tfrac{9}{4}\times 60+12\times\tfrac{11}{2}\times\tfrac{10}{3}\times\tfrac{9}{4}\times\tfrac{8}{5}\times 24.$$

Reducing this expression,

$$S=12+990+11000+29700+19008=60710.$$

* * *

## CONTINUED FRACTIONS.

ART. 528. Ex. 1. What is the continued fraction formed from $\frac{235}{1683}$?

235 ) 1683 ( 7 $\frac{1}{7}$ = 1st Integral Fraction.
1645
1st Rem. 38 ) 235 ( 6 $\frac{1}{6}$ = 2d " "
228
2d Rem. 7 ) 38 ( 5 $\frac{1}{5}$ = 3d " "
35
3d Rem. 3 ) 7 ( 2 $\frac{1}{2}$ = 4th " "
6
4th Rem. 1 ) 3 ( 3 $\frac{1}{3}$ = 5th " "
3
0

The continued fraction is formed by adding the 2d integral fraction to the denominator of the 1st, the 3d to the denominator of the 2d, and so on.

Ex. 2. What is the continued fraction formed from $\frac{1103}{887}$?

887 ) 1103 ( 1
887
1st Rem. 216 ) 887 ( 4 $1+\frac{1}{4}$ = 1st Integ. Fract.
864
2d Rem. 23 ) 216 ( 9 $\frac{1}{9}$ = 2d " "
207
3d Rem. 9 ) 23 ( 2 $\frac{1}{2}$ = 3d " "
18
4th Rem. 5 ) 9 ( 1 $\frac{1}{1}$ = 4th " "
5
5th Rem. 4 ) 5 ( 1 $\frac{1}{1}$ = 5th " "
4
6th Rem. 1 ) 4 ( 4 $\frac{1}{4}$ = 6th " "
4
0

Ex. 3. What are the integral fractions derived from $\frac{1461}{59}$?

59) 1461 (24
118
281
236
1st Rem. 45) 59 (1 $24+\frac{1}{1}$=1st Integral Fraction.
45
2d Rem. 14) 45 (3 $\frac{1}{3}$=2d " "
42
3d Rem. 3) 14 (4 $\frac{1}{4}$=3d " "
12
4th Rem. 2) 3 (1 $\frac{1}{1}$=4th " "
2
5th Rem. 1) 2 (2 $\frac{1}{2}$=5th " "
2
0

Art. 529. Ex. Reduce the $\sqrt{2}$ to the form of a continued fraction.

1000000) 1414213 (1
1000000
1st Rem. 414213) 1000000 (2
828426
2d Rem. 171574) 414213 (2
343148
3d Rem. 71065) 171574 (2
142130
4th Rem. 29444

The first term of the continued fraction is 1, and each succeeding term is $\frac{1}{2}$.

## INTERPOLATION.

Ex. 2. Given the logarithmic sines of 20°, 22°, 24°, &c.: required the sine of $23\frac{1}{2}$°.

Here the interval between the numbers is 2. The distance $m$ of the required sine from the first number is $1\frac{3}{4}$ of this interval.

Therefore, $mD'=\frac{7}{4}D'$, $\qquad m\frac{(m-1)}{2}D''=\frac{7}{4}\times\frac{3}{8}D''$,

$$m\frac{(m-1)}{2}\frac{(m-2)}{3}D'''=\frac{7}{4}\times\frac{3}{8}\times-\frac{1}{12}D''',$$

$$m\frac{(m-1)}{2}\frac{(m-2)}{3}\frac{(m-3)}{4}D''''=\frac{7}{4}\times\frac{3}{8}\times-\frac{1}{12}\times-\frac{5}{16}D''''.$$

The logarithmic sines given in the tables are

| | | | | | |
|---|---|---|---|---|---|
| For 20° | 534052 | $D'=39523$ | $D''=-3785$ | $D'''=576$ | $D''''=-129.$ |
| " 22° | 573575 | 35738 | $-3209$ | 447 | |
| " 24° | 609313 | 32529 | $-2762$ | | |
| " 26° | 641842 | 29767 | | | |
| " 28° | 671609 | | | | |

The first term $a$ $\qquad =534052$

$$mD'=39523\times\frac{7}{4} \qquad = \ 69165$$

$$m\frac{(m-1)}{2}D''=-3785\times\frac{7}{4}\times\frac{3}{8} \qquad =-2484$$

$$m\frac{(m-1)}{2}\frac{(m-2)}{3}D'''=576\times\frac{7}{4}\times\frac{3}{8}\times-\frac{1}{12} \qquad = \ -31$$

$$m\frac{(m-1)}{2}\frac{(m-2)}{3}\frac{(m-3)}{4}D''''=-129\times\frac{7}{4}\times\frac{3}{8}\times-\frac{1}{12}\times-\frac{5}{16} \qquad = \ -2$$

The required sine, $\qquad =600700$

Ex. 3. Given the logarithmic tangents of 17°, 18°, 19°, 20°: required the tangent of $18\frac{1}{2}$°.

Here the interval between the numbers is 1. The distance $m$ of the required tangent from the first number is $1\frac{1}{2}$ of this interval.

Therefore, $mD'=\frac{3}{2}D'$, $\qquad m\frac{(m-1)}{2}D''=\frac{3}{2}\times\frac{1}{4}D''$,

$$m\frac{(m-1)}{2}\frac{(m-2)}{3}D'''=\frac{3}{2}\times\frac{1}{4}\times-\frac{1}{6}D'''.$$

The logarithmic tangents given in the tables are

| | | | | |
|---|---|---|---|---|
| For 17° | 485339 | $D'=26437$ | $D''=-1241$ | $D'''=139.$ |
| " 18° | 511776 | 25196 | $-1102$ | |
| " 19° | 536972 | 24094 | | |
| " 20° | 561066 | | | |

The first term $a$ $=485339$

The second $mD'=26437\times\frac{3}{2}$ $=39655$

The third $m\frac{(m-1)}{2}D''=-1241\times\frac{3}{2}\times\frac{1}{4}$ $=-465$

The fourth $m\frac{(m-1)}{2}\frac{(m-2)}{3}D'''=139\times\frac{3}{2}\times\frac{1}{4}\times-\frac{1}{6}$ $=-9$

The required tangent $=524520.$

Ex. 4. Given the natural sines of 24°, 25°, 26°, 27°: required the natural sine of $25\frac{2}{3}°$.

Here the interval between the numbers is 1. The distance $m$ of the required sine from the first number is $1\frac{2}{3}$ of this interval.

Therefore, $mD'=\frac{5}{3}\times D'$, $m\frac{(m-1)}{2}D''=\frac{5}{3}\times\frac{2}{6}\times D''$.

The natural sines given in the tables are

| | | | | |
|---|---|---|---|---|
| For 24° | 40674 | $D'=1588$ | $D''=-13$ | $D'''=0.$ |
| " 25° | 42262 | 1575 | $-13$ | |
| " 26° | 43837 | 1562 | | |
| " 27° | 45399 | | | |

The first term $a$ $=40674$

The second $mD'=1588\times\frac{5}{3}$ $=2646$

The third $m\frac{(m-1)}{2}D''=-13\times\frac{5}{3}\times\frac{2}{6}$ $=-7$

The required sine $=43313.$

## GENERAL PROPERTIES OF EQUATIONS.

### ARTICLE 537. EXAMPLE 1.

Prove that 2 is a root of the equation $x^3-7x^2+12x-4=0$.

$$\begin{array}{r|l}
x^3-7x^2+12x-4 & x-2 \\ \cline{2-2}
\underline{x^3-2x^2\phantom{+12x-4}} & x^2-5x+2 \\
-5x^2+12x\phantom{-4} & \\
\underline{-5x^2+10x\phantom{-4}} & \\
2x-4 & \\
\underline{2x-4} & \\
\end{array}$$

There is no remainder, and hence 2 is a root of the equation.

### Example 2.

Prove that 3 is not a root of $x^3+2x^2-6x-4=0$.

$$\begin{array}{r|l}
x^3+2x^2-6x-4 & x-3 \\ \cline{2-2}
\underline{x^3-3x^2\phantom{-6x-4}} & x^2+5x+9 \\
5x^2-\phantom{1}6x\phantom{-4} & \\
\underline{5x^2-15x\phantom{-4}} & \\
9x-\phantom{2}4 & \\
\underline{9x-27} & \\
23=\text{Rem.} & \\
\end{array}$$

There is a remainder, and hence 3 is not a root of the equation.

### Example 3.

Find whether $-1$ is a root of $x^4-x^3-5x^2+4x-3=0$.

$$\begin{array}{r|l}
x^4-x^3-5x^2+4x-3 & x+1 \\ \cline{2-2}
\underline{x^4+x^3\phantom{-5x^2+4x-3}} & x^3-2x^2-3x+7 \\
-2x^3-5x^2\phantom{+4x-3} & \\
\underline{-2x^3-2x^2\phantom{+4x-3}} & \\
-3x^2+4x\phantom{-3} & \\
\underline{-3x^2-3x\phantom{-3}} & \\
7x-\phantom{1}3 & \\
\underline{7x+\phantom{1}7} & \\
-10=\text{Rem.} & \\
\end{array}$$

There is a remainder, and hence $-1$ is not a root of the equation.

### EXAMPLE 4.

Find whether 1 is a root of $x^4-2x^3-11x^2-8x+15=0$.

$$\begin{array}{r|l}
x^4-2x^3-11x^2-8x+15 & x-1 \\ \cline{2-2}
x^4-x^3\phantom{-11x^2-8x+15} & x^3-x^2-12x-20 \\ \cline{1-1}
-x^3-11x^2\phantom{-8x+15} & \\
-x^3+x^2\phantom{-8x+15} & \\ \cline{1-1}
-12x^2-8x\phantom{+15} & \\
-12x^2+12x\phantom{+15} & \\ \cline{1-1}
-20x+15 & \\
-20x+20 & \\ \cline{1-1}
-5=\text{Rem.} &
\end{array}$$

There is a remainder, and hence 1 is not a root of the equation.

### EXAMPLE 5.

Find whether $-5$ is a root of $x^5+3x^3-64x^2+x+23=0$.

$$\begin{array}{r|l}
x^5+3x^3-64x^2+x+23 & x+5 \\ \cline{2-2}
x^5+5x^4\phantom{-64x^2+x+23} & x^4-5x^3+28x^2-204x+1021 \\ \cline{1-1}
-5x^4+3x^3\phantom{-64x^2+x+23} & \\
-5x^4-25x^3\phantom{-64x^2+x+23} & \\ \cline{1-1}
28x^3-64x^2\phantom{+x+23} & \\
28x^3+140x^2\phantom{+x+23} & \\ \cline{1-1}
-204x^2+x\phantom{+23} & \\
-204x^2-1020x\phantom{+23} & \\ \cline{1-1}
1021x+23 & \\
1021x+5105 & \\ \cline{1-1}
-5082=\text{Rem.} &
\end{array}$$

There is a remainder, and hence $-5$ is not a root of the equation.

### ARTICLE 538. EXAMPLE 1.

One root of the equation $x^3-4x^2+2x+3=0$ is 3. What are the other roots?

$$\begin{array}{r|l}
x^3-4x^2+2x+3 & x-3 \\ \cline{2-2}
\underline{x^3-3x^2\phantom{+2x+3}} & x^2-x-1 \\
-x^2+2x\phantom{+3} & \\
\underline{-x^2+3x\phantom{+3}} & \\
-x+3 & \\
\underline{-x+3} &
\end{array}$$

The equation $x^2-x-1=0$ contains the two required roots.

To solve this equation,

Transposing and comp. the square, $x^2-x+\frac{1}{4}=\frac{1}{4}+1=\frac{5}{4}$

Extracting the square root, $x-\frac{1}{2}=\frac{\pm\sqrt{5}}{2}$, or $x=\frac{1\pm\sqrt{5}}{2}$

Therefore the required roots are $\frac{1+\sqrt{5}}{2}$ and $\frac{1-\sqrt{5}}{2}$.

Ex. 2. One root of the equation $x^3-5x^2+3x+9=0$ is 3. What are the other two?

$$\begin{array}{r|l}
x^3-5x^2+3x+9 & x-3 \\ \cline{2-2}
\underline{x^3-3x^2\phantom{+3x+9}} & x^2-2x-3 \\
-2x^2+3x\phantom{+9} & \\
\underline{-2x^2+6x\phantom{+9}} & \\
-3x+9 & \\
\underline{-3x+9} &
\end{array}$$

The equation $x^2-2x-3=0$ contains the two required roots.

Transposing and completing the square, $x^2-2x+1=4$

Extracting the square root, $x-1=\pm2$, or $x=1\pm2$

Therefore the required roots are 3 and $-1$.

Ex. 3. One root of the equation $x^3+5x^2-3x-7=0$ is $-1$. What are the other two?

$$\begin{array}{r|l}
x^3+5x^2-3x-7 & x+1 \\ \cline{2-2}
\underline{x^3+\phantom{5}x^2\phantom{-3x-7}} & x^2+4x-7 \\
4x^2-3x\phantom{-7} & \\
\underline{4x^2+4x\phantom{-7}} & \\
-7x-7 & \\
\underline{-7x-7} &
\end{array}$$

The equation $x^2+4x-7=0$ contains the required roots.
Transposing and comp. the square, $x^2+4x+4=7+4=11$
Extracting the square root, $x+2=\pm\sqrt{11}$, or $x=-2\pm\sqrt{11}$
Therefore the required roots are $-2+\sqrt{11}$, and $-2-\sqrt{11}$.

Ex. 4. Two roots of the equation $x^4-x^3+2x^2-14x+12=0$ are 1 and 2. What are the other roots?

$$\begin{array}{l|l}
x^4-\ x^3+2x^2-14x+12 & x-2=\text{1st Div.} \\ \cline{2-2}
\underline{x^4-2x^3} & x^3+x^2+4x-6=\text{1st Quot.} \\
\quad x^3+2x^2 & \\
\quad \underline{x^3-2x^2} & \\
\qquad 4x^2-14x & \\
\qquad \underline{4x^2-\ 8x} & \\
\qquad\quad -\ 6x+12 & \\
\qquad\quad \underline{-\ 6x+12} &
\end{array}$$

$$\begin{array}{l|l}
x^3+x^2+4x-6 & x-1=\text{2d Div.} \\ \cline{2-2}
\underline{x^3-x^2} & x^2+2x+6=\text{2d Quot.} \\
\quad 2x^2+4x & \\
\quad \underline{2x^2-2x} & \\
\qquad 6x-6 & \\
\qquad \underline{6x-6} &
\end{array}$$

The equation $x^2+2x+6=0$ contains the required roots.
Trans. and comp. the square, $x^2+2x+1=-6+1=-5$
Extract. the square root, $x+1=\pm\sqrt{-5}$, or $x=-1\pm\sqrt{-5}$
Therefore the required roots are $-1+\sqrt{-5}$ and $-1-\sqrt{-5}$.

Ex. 5. Two roots of the equation $x^4+x^3-7x^2-x+6=0$ are 1 and $-3$. What are the other roots?

$$\begin{array}{l|l}
x^4+\ x^3-7x^2-x+6 & x^2+2x-3 \\ \cline{2-2}
\underline{x^4+2x^3-3x^2} & x^2-x-2 \\
\quad -\ x^3-4x^2-\ x & \\
\quad \underline{-\ x^3-2x^2+3x} & \\
\qquad -2x^3-4x+6 & \\
\qquad \underline{-2x^2-4x+6} &
\end{array}$$

$(x-1)\,(x+3)=x^2+2x-3$ is the divisor; and the quotient placed equal to zero is the equation which contains the two required roots. In this equation

Transposing and comp. the square, $x^2-x+\frac{1}{4}=\frac{1}{4}+2=\frac{9}{4}$

Extracting the square root, $x-\frac{1}{2}=\pm\frac{3}{2}$, or $x=\frac{1\pm3}{2}$

Therefore the required roots are 2 and $-1$.

Ex. 6. One root of the equation $x^4-2x^3+x^2-5x-9=0$ is $-1$. What is the equation containing the other roots?

$$\begin{array}{rl}
x^4-2x^3+x^2-5x-9 & \underline{|\,x+1\qquad\qquad\quad} \\
\underline{x^4+\ x^3\qquad\qquad\quad} & x^3-3x^2+4x-9 \\
-3x^3+\ x^2\qquad\quad & \\
\underline{-3x^3-3x^2\qquad\quad} & \\
4x^2-5x\quad & \\
\underline{4x^2+4x\quad} & \\
-9x-9 & \\
\underline{-9x-9} &
\end{array}$$

The equation $x^3-3x^2+4x-9=0$ contains the other roots.

ART. 542. Ex. 1. Form the equation whose roots are 1, $-1$, 2, $-3$.

Since there are 4 roots the equation is biquadratic, and the law of coefficients (Art. 541) gives,

$$x^4\quad \left.\begin{array}{r}-1\\+1\\-2\\+3\end{array}\right\}x^3\quad \left.\begin{array}{r}-1\\+2\\-3\\-2\\+3\\-6\end{array}\right\}x^2\quad \left.\begin{array}{r}+2\\-3\\+6\\-6\end{array}\right\}x+6=0$$

Or by reduction, $x^4+x^3-7x^2-x+6=0$

The same equation may be formed by multiplying together the factors $(x-1)$, $(x+1)$, $(x-2)$, $(x+3)$, as follows: $(x-1)\times(x+1)=x^2-1$ (Art. 111).

$$\begin{array}{rrrr}
x^2 & & -1 & \\
x & -2 & & \\
\hline
x^3 & & -x & \\
 & -2x^2 & & +2 \\
\hline
x^3 & -2x^2 & -x & +2
\end{array}
\qquad
\begin{array}{rrrrr}
x^3 & -2x^2 & -x & +2 & \\
 & & x & +3 & \\
\hline
x^4 & -2x^3 & -x^2 & +2x & \\
 & +3x^3 & -6x^2 & -3x & +6 \\
\hline
x^4 & +x^3 & -7x^2 & -x & +6=0.
\end{array}$$ *Ans.*

Ex. 2. Form the equation whose roots are 1, 2, $-2$, 3.

Since there are 4 roots the equation is biquadratic, and the law of coefficients (Art. 541) gives,

$$x^4 \quad \left.\begin{matrix}-1\\-2\\+2\\-3\end{matrix}\right\} x^3 \quad \left.\begin{matrix}+2\\-2\\+3\\-4\\+6\\-6\end{matrix}\right\} x^2 \quad \left.\begin{matrix}+\ 4\\-\ 6\\+\ 6\\+12\end{matrix}\right\} x-12=0$$

Or by reduction, $\quad x^4-4x^3-x^2+16x-12=0.$

This equation may be formed also by multiplying together the factors $(x-1)$, $(x-2)$, $(x+2)$, $(x-3)$, as follows: $(x-2)\times(x+2)=x^2-4$ (Art. 111).

$$\begin{array}{rrrr}
x^2 & & -4 & \\
x & -3 & & \\
\hline
x^3 & & -4x & \\
 & -3x^2 & & +12 \\
\hline
x^3 & -3x^2 & -4x & +12
\end{array}
\qquad
\begin{array}{rrrrr}
x^3 & -3x^2 & -4x & +12 & \\
 & & x & -\ 1 & \\
\hline
x^4 & -3x^3 & -4x^2 & +12x & \\
 & -\ x^3 & +3x^2 & +\ 4x & -12 \\
\hline
x^4 & -4x^3 & -\ x^2 & +16x & -12=0,
\end{array}$$ *Ans.*

Ex. 3. Form the equation whose roots are $-1$, 2, $-3$.

Since there are 3 roots the equation is a cubic equation, and the law of coefficients (Art. 541) gives,

$$x^3 \quad \left.\begin{matrix}+1\\-2\\+3\end{matrix}\right\} x^2 \quad \left.\begin{matrix}-2\\+3\\-6\end{matrix}\right\} x \quad -6=0$$

Or by reduction, $\quad x^3+2x^2-5x-6=0.$

Ex. 4. Form the equation whose roots are 1, 2, 2, −2, −3.

Since there are 5 roots the equation is of the fifth degree, and the law of coefficients (Art. 541) gives,

$$x^5 \left.\begin{array}{r}-1\\-2\\-2\\+2\\+3\end{array}\right\} x^4 \left.\begin{array}{r}+2\\+2\\-2\\-3\\+4\\-4\\-6\\-4\\-6\\+6\end{array}\right\} x^3 \left.\begin{array}{r}-\ 4\\+\ 4\\+\ 6\\+\ 4\\+\ 6\\-\ 6\\+\ 8\\+12\\-12\\-12\end{array}\right\} x^2 \left.\begin{array}{r}-\ 8\\-12\\+12\\+12\\+24\end{array}\right\} x-24=0$$

Or by reduction, $x^5-11x^3+6x^2+28x-24=0.$

Art. 544. Ex. 2. Required the terms of the biquadratic equation in which $S_1=1$, $S_2=39$, $S_3=-89$, and the product of all the roots, after their signs are changed, is −30.

$$A=-1, \qquad B=-\frac{1}{2}(-1+39)=-\frac{38}{2}=-19,$$

$$C=-\frac{1}{3}(-19-39-89)=-\frac{1}{3}(-147)=49.$$

Therefore the required equation is $x^4-x^3-19x^2+49x-30=0.$

Art. 545. Ex. 1. Find the successive derived polynomials of $r^3-r^2+2r+6$.

Multiply each term by the exponent of $r$ in that term, and diminish the exponent by unity.

$$R'=3r^2-2r+2, \qquad R''=6r-2, \qquad R'''=6.$$

Ex. 2. Find the derived polynomials of $z^4+2z^2-6z-4$.

$$R'=4z^3+4z-6, \qquad R'''=24z,$$
$$R''=12z^2+4, \qquad R''''=24.$$

Ex. 3. Find the derived polynomials of $x^5-3x^4-3x^2+5$.

$$R'=5x^4-12x^3-6x, \qquad R'''=60x^2-72x,$$
$$R''=20x^3-36x^2-6, \qquad R''''=120x-72,$$
$$R'''''=120.$$

Ex. 4. Find the derived polynomials of $x^4-x^3+2x^2-3x+4$.

$$R'=4x^3-3x^2+4x-3, \qquad R'''=24x-6,$$
$$R''=12x^2-6x+4, \qquad R''''=24.$$

Art. 546. Ex. 3. Find the equation whose roots are greater by 2 than those of the equation $x^4-2x^3+5x^2+4x-8=0$.

Put $y=x+r$. The condition $y=x-r$, or $x=r+y$ would give

$$R=r^4-2r^3+5r^2+4r-8, \qquad R''=12r^2-12r+10,$$
$$R'=4r^3-6r^2+10r+4, \qquad R'''=24r-12,$$
$$R''''=24.$$

And the given equation would be

$$R+R'y+R''\frac{y^2}{2}+R'''\frac{y^3}{2.3}+R''''\frac{y^4}{2.3.4}=0.$$

But this will also represent the required equation, if the sign of $r$ is regarded as negative—that is, if the signs of the odd powers of $r$ in the values of $R$, $R'$, &c., be changed. Making this change, and substituting for $R$, $R'$, &c., their values,

$$(r^4+2r^3+5r^2-4r-8)+(-4r^3-6r^2-10r+4)y+$$
$$(12r^2+12r+10)\frac{y^2}{2}+(-24r-12)\frac{y^3}{2.3}+24\frac{y^4}{2.3.4}=0.$$

But $r=2$. Substituting for $r$ its value,

$$(16+16+20-8-8)+(-32-24-20+4)y+$$
$$(48+24+10)\frac{y^2}{2}+(-48-12)\frac{y^3}{2.3}+24\frac{y^4}{2.3.4}=0.$$

Reducing and writing in the usual way,

$$y^4-10y^3+41y^2-72y+36=0.$$

Ex. 4. Find the equation whose roots are less by 1 than those of the equation $x^5+3x^3-7x^2+x+12=0$.

Put $y=x-r$. Then $x=r+y$, and by substitution the given equation will become

$$R+R'y+R''\frac{y^2}{2}+R'''\frac{y^3}{2.3}+R''''\frac{y^4}{2.3.4}, \text{\&c.}=0.$$

But, by Art. 545,

$$\begin{aligned} R\ &=r^5+3r^3-7r^2+r+12, & R'''\ &=60r^2+18,\\ R'\ &=5r^4+9r^2-14r+1, & R''''\ &=120r,\\ R''&=20r^3+18r-14, & R'''''&=120. \end{aligned}$$

Substituting these values of $R$, $R'$, &c., in the above equation,

$$(r^5+3r^3-7r^2+r+12)+(5r^4+9r^2-14r+1)y+(20r^3+$$

$$18r-14)\frac{y^2}{2}+(60r^2+18)\frac{y^3}{2.3}+120r\frac{y^4}{2.3.4}+120\frac{y^5}{2.3.4.5}=0.$$

But $r=1$. Subst. and writing the equation in the usual way,

$$y^5+5y^4+13y^3+12y^2+y+10=0.$$

Ex. 5. Find the equation whose roots are greater by 1 than those of the equation $x^4-5x^2-6x-2=0$.

Put $y=x+r$. The condition $y=x-r$, or $x=y+r$ would give for the derived polynomials,

$$\begin{aligned} R\ &=r^4-5r^2-6r-2, & R''\ &=12r^2-10,\\ R'&=4r^3-10r-6, & R'''&=24r, \end{aligned}$$

$$R''''=24.$$

And the given equation would become

$$R+R'y+R''\frac{y^2}{2}+R'''\frac{y^3}{2.3}+R''''\frac{y^4}{2.3.4}=0.$$

But this will also represent the required equation, if the sign of $r$ is regarded as negative—that is, if the signs of the odd powers of $r$, in the values of $R$, $R'$, &c., be changed. Making this change, and substituting for $R$, $R'$, &c., their values,

$$(r^4-5r^2+6r-2)+(-4r^3+10r-6)y+(12r^2-10)$$

$$\frac{y^2}{2}+24r\frac{y^3}{2.3}+24\frac{y^4}{2.3.4}=0.$$

But $r=1$. Substituting this value of $r$, and reducing and writing the equation in the usual way,

$$y^4-4y^3+y^2=0.$$

Since the known term in this last equation equals zero, at least one of the roots must be equal to zero: for the known term equals the product of all the roots after their signs have been changed. Then it may be shown, as in the Algebra, that the roots of the original equation are $-1$, $-1$, $1+\sqrt{3}$, $1-\sqrt{3}$.

ART. 547. Ex. 3. Divide $2x^4-5x^2-14x-20$ by $x-1$.

$$\begin{array}{ccccc}
A & B & C & D & E \\
2 & 0 & -5 & -14 & -20 \\
 & 2 & 2 & -3 & -17 \\
\hline
 & 2 & -3 & -17 & -37 \\
 & B' & C' & D' & E'
\end{array}$$

Substituting these values of $A$, $B'$, $C'$, &c., in the expression for the quotient of the general equation of the fourth degree divided by $x-a$, which is

$$Ax^3+B'x^2+C'x+D'+\frac{E'}{x-a},$$

We have, $$2x^3+2x^2-3x-17-\frac{37}{x-1}.$$

Ex. 4. Divide $5x^3-5x^2+10x-12$ by $x+2$.

$$\begin{array}{cccc}
5 & -5 & 10 & -12 \\
 & 10 & -30 & 80 \\
\hline
 & -15 & 40 & -92 \\
A & B' & C' & D'
\end{array}$$

The general expression for the quotient of an equation of the third degree divided by $x+2$ is $Ax^2+B'x+C'+\dfrac{D'}{x+a}$.

Substituting for $a$, $A$, $B'$, &c., their values, we have,

$$5x^2-15x+40-\frac{92}{x+2}.$$

Ex. 5. Divide $x^5-5x^4-6x^3-x+1$ by $x+1$.

$$\begin{array}{cccccc}
1 & -5 & -6 & 0 & -1 & 1 \\
 & 1 & -6 & 0 & 0 & -1 \\
\hline
 & -6 & 0 & 0 & -1 & 2 \\
A & B' & C' & D' & E' & F'
\end{array}$$

The general expression for the quotient of an equation of the fifth degree divided by $x+a$ is

$$Ax^4+B'x^3+C'x^2+D'x+E'+\frac{F'}{x+a}.$$

Substituting for $a$, $A$, $B'$, &c., their values, we have,

$$x^4-6x^3-1+\frac{2}{x+1}.$$

ART. 548. Ex. 4. Diminish by .02 the roots of the equation $2x^4-x^3-10x+12=0$.

Here the coefficients are 2, −1, 0, −10, 12; and the process is as follows:

$$\begin{array}{rrrrl}
2 & -1 & 0 & -10 & 12 \\
 & .04 & -.0192 & -\ \ .000384 & -.20000768 \\
\cline{2-5}
 & -\ \ .96 & -.0192 & -10.000384 & 11.79999232=\text{1st Rem.} \\
 & .04 & -.0184 & .000752 & \\
\cline{2-4}
 & -\ \ .92 & -.0376 & -10.001136=\text{2d Rem.} & \\
 & .04 & -.0176 & & \\
\cline{2-3}
 & -\ \ .88 & -.0552=\text{3d Rem.} & & \\
 & .04 & & & \\
\cline{2-2}
 & -\ \ .84=\text{4th Rem.} & & &
\end{array}$$

Using the several remainders as the Rule directs, the required equation is found to be,

$$2y^4-.84y^3-.0552y^2-10.001136y+11.79999232=0.$$

Ex. 5. Diminish by .03 the roots of the equation

$$4x^3-.36x^2+.08x+2.15=0.$$

Here the coefficients are 4, −.36, .08, 2.15; and the process is as follows:

$$\begin{array}{rrrl}
4 & -.36 & .08 & 2.15 \\
 & .12 & -.0072 & .002184 \\
\cline{2-4}
 & -.24 & .0728 & 2.152184=\text{1st Rem.} \\
 & .12 & -.0036 & \\
\cline{2-3}
 & -.12 & .0692=\text{2d Rem.} & \\
 & .12 & & \\
\cline{2-2}
 & 0\ =\text{3d Rem.} & &
\end{array}$$

Arranging the several remainders, which are the coefficients of the required equation, as the Rule directs, and the result is

$$4y^3+.0692y+2.152184=0.$$

### ARTICLE 549. EXAMPLE 1.

Transform $x^3-6x^2+7=0$ into an equation whose second term is wanting.

$$\text{Let } x=y-\frac{A}{m}=y+\frac{6}{3}=y+2.$$

Subst. this value of $x$ in the given equation,

$$(y+2)^3-6(y+2)^2+7=0$$

Expanding, $y^3+6y^2+12y+8-6y^2-24y-24+7=0$

Reducing, $y^3-12y-9=0.$

### EXAMPLE 2.

Transform $x^3+3x^2-3x-1=0$ into an equation whose second term is wanting.

$$\text{Let } x=y-\frac{A}{m}=y-\frac{3}{3}=y-1.$$

Subst. for $x$ in the given equation,

$$(y-1)^3+3(y-1)^2-3(y-1)-1=0$$

Expanding, $y^3-3y^2+3y-1+3y^2-6y+3-3y+3-1=0$

Reducing, $y^3-6y+4=0.$

### EXAMPLE 3.

Transform $x^4-8x^3-5x+12=0$ into an equation whose second term is wanting.

$$\text{Let } x=y-\frac{A}{m}=y+\frac{8}{4}=y+2.$$

Subst. for $x$ in the given equation,

$$(y+2)^4-8(y+2)^3-5(y+2)+12=0$$

Expanding, $y^4+8y^3+24y^2+32y+$
$16-8y^3-48y^2-96y-64-5y-10+12=0$

Reducing, $y^4-24y^2-69y-46=0.$

### EXAMPLE 4.

Transform $x^5+20x^4+2x^2-3x-24=0$ into an equation whose second term is wanting.

$$\text{Let } x=y-\frac{A}{m}=y-\frac{20}{5}=y-4.$$

Substituting,

$$(y-4)^5+20(y-4)^4+2(y-4)^2-3(y-4)-24=0$$

Expanding,

$$y^5-20y^4+160y^3-640y^2+1280y-1024+20y^4-320y^3+$$
$$1920y^2-5120y+5120+2y^2-16y+32-3y+12-24=0$$

Reducing, $y^5-160y^3+1282y^2-3859y+4116=0.$

### ARTICLE 550. EXAMPLE 1.

The roots of the equation $x^3+4x^2+x-6=0$ are 1, $-2$, $-3$. What are those of the equation $x^3-4x^2+x+6=0$?

Changing the signs of the given roots, we have $-1$, 2, 3, which are the roots of the second equation.

*Proof.* By Art. 541,

$$x^3 \quad \left.\begin{matrix}+1\\-2\\-3\end{matrix}\right\} x^2 \qquad \left.\begin{matrix}-2\\-3\\+6\end{matrix}\right\} x \quad +6=0$$

Or by reduction, $x^3-4x^2+x+6=0,$

Which is the second given equation; and hence $-1$, 2, 3 are its roots.

### EXAMPLE 2.

The roots of the equation $x^4-4x^3-7x^2+34x-24=0$ are 1, 2, $-3$, 4. What are those of the equation

$$x^4+4x^3-7x^2-34x-24=0?$$

Changing the signs of the given roots, we have $-1$, $-2$, 3, $-4$, which are the roots of the second equation.

*Proof.* Multiply together the factors $(x+1)$, $(x+2)$, $(x-3)$, $(x+4)$, and the product will be the second given equation, if $-1$, $-2$, 3, $-4$, are its roots.

$$\begin{array}{rrrrr}
 & & & x & +1 \\
 & & & x & +2 \\ \hline
 & & x^2 & +x & \\
 & & & +2x & +2 \\ \hline
 & & x^2 & +3x & +2 \\
 & & & x & -3 \\ \hline
 & x^3 & +3x^2 & +2x & \\
 & & -3x^2 & -9x & -6 \\ \hline
 & x^3 & & -7x & -6 \\
 & & & x & +4 \\ \hline
x^4 & & -7x^2 & -6x & \\
 & +4x^3 & & -28x & -24 \\ \hline
x^4 & +4x^3 & -7x^2 & -34x & -24=0
\end{array}$$

The last product is the second given equation, and hence $-1$, $-2$, $3$, $-4$ are its roots.

Ex. 3. The roots of the equation $x^4+4x^3+3x^2-4x-4=0$ are $1$, $-1$, $-2$, $-2$. What are those of the equation

$$x^4-4x^3+3x^2+4x-4=0\,?$$

Changing the signs of the given roots, we have $-1$, $1$, $2$, $2$, which are the roots of the second given equation.

*Proof.* Multiply together the factors $(x+1)$, $(x-1)$, $(x-2)$, $(x-2)$, and the product will be the second given equation, if $-1$, $1$, $2$, $2$, are its roots.

Ex. 4. The roots of the equation $x^3-7x+6=0$ are $1$, $2$, $-3$. What is the equation whose roots are $-1$, $-2$, $3$?

Change the sign of the known term in the given equation, and the result will be the required equation.

*Proof.* If this result is the required equation, it will be equal to the product of $(x+1)\times(x+2)\times(x-3)$.

ART. 552. Ex. 2. What value of $x$ will make the first term of the polynomial $x^3-11x^2-10x-9$ greater than the sum of all the other terms?

Here the largest coefficient is 11. Increasing this by 1, we have 12 for the required value of $x$.

Ex. 3. What value of $x$ will make the first term of the polynomial $x^4+7x^3+6x^2-7x+5$ greater than the sum of all the rest?

Here the largest coefficient is 7. Increasing this by 1, we have 8 for the required value of $x$.

ART. 553. Ex. 1. What is the integral part of a root of the equation $x^3-7x^2+3x-4=0$?

The proper course to be taken in problems of this sort, unless we can at once approximate to the result by inspection, is to suppose various values of $x$ and substitute them in the given equation, until two contiguous numbers are found which give results with contrary signs. One root at least will be comprised between these numbers.

Let $x=1$ Then, $1-7+3-4<0$
" $x=3$ " $8-28+6-4<0$
" $x=5$ " $125-175+15-4<0$
" $x=7$ " $343-343+21-4>0$
" $x=6$ " $216-252+18-4<0$

The numbers 6 and 7 when substituted for $x$ give results with contrary signs. Hence there is a root comprised between them, and its integral part is 6.

Ex. 2. Find the integral parts of two roots of the equation

$$x^4-12x^3-20x^2-36=0.$$

Let $x=1$ Then $1-12-20-36<0$
" $x=3$ " $81-324-180-36<0$
" $x=5$ " $625-1500-500-36<0$
" $x=7$ " $2401-4116-980-36<0$
" $x=9$ " $6561-8748-1620-36<0$
" $x=11$ " $14641-15972-2420-36<0$
" $x=13$ " $28561-26364-3380-36<0$
" $x=14$ " $38416-32928-3920-36>0$

The numbers 13 and 14 when substituted for $x$ give results with contrary signs. Hence there is a root of the equation comprised between them, and its integral part is 13.

Let $x=-1$ Then $1+12-20-36<0$
“ $x=-2$ “ $16+96-80-36<0$
“ $x=-3$ “ $81+324-180-36>0$

Therefore there is a root of the equation comprised between the numbers $-2$ and $-3$, and its integral part is $-2$.

Ex. 3. Find the integral parts of two roots of the equation,

$$x^4-4x^3+5x^2-2x-6=0$$

Let $x=1$ Then $1-4+5-2-6<0$
“ $x=-1$ “ $1+4+5+2-6>0$

Therefore there is a root between 1 and $-1$, and its integral part must be 0.

Let $x=2$ Then $16-32+20-4-6<0$
“ $x=3$ “ $81-108+45-6-6>0$

Hence there is a root between 2 and 3, and its integral part is 2.

Ex. 4. Find the integral part of a root of the equation

$$x^3-x^2+7=0.$$

Let $x=-1$ Then $-1-1+7>0$
“ $x=-2$ “ $-8-4+7<0$

Hence there is a root of the equation between $-1$ and $-2$, and its integral part is $-1$.

Art. 558. Ex. 1. Find the cubic equation of which two of the roots are 1 and $3+\sqrt{-2}$.

The other root must be $3-\sqrt{-2}$, and the required equation equals the product of the factors

$$(x-3-\sqrt{-2}),\ (x-3+\sqrt{-2}),\ (x-1).$$

Now $x-3-\sqrt{-2}=x-3-\sqrt{2}\times\sqrt{-1}$
And $x-3+\sqrt{-2}=x-3+\sqrt{2}\times\sqrt{-1}$

$$\begin{array}{l}
x-3 \quad - \quad \sqrt{2}\times\sqrt{-1} \\
x-3 \quad + \quad \sqrt{2}\times\sqrt{-1} \\
\hline
x^2-3x-x\sqrt{2}\times\sqrt{-1} \\
\quad -3x \qquad\qquad\qquad +3\sqrt{2}\times\sqrt{-1}+9 \\
\qquad\quad +x\sqrt{2}\times\sqrt{-1}-3\sqrt{2}\times\sqrt{-1}-2\times-1 \\
\hline
x^2-6x+11 \\
\qquad x-\ 1 \\
\hline
x^3-6x^2+11x \\
\quad -\ x^2+\ 6x-11 \\
\hline
x^3-7x^2+17x-11=0=\text{The required equation.}
\end{array}$$

Ex. 2. One root of the equation $x^3-x^2+2=0$ is $-1$. What are the other roots?

Divide the given equation by $(x+1)$, and the quotient will contain the two required roots.

$$\begin{array}{l|l}
x^3-\ x^2+2 & x+1 \\
\cline{2-2}
x^3+\ x^2 & x^2-2x+2=0. \quad \text{This contains the other roots.} \\
\cline{1-1}
\quad -2x^2+2 & \\
\quad -2x^2-2x & \\
\cline{1-1}
\qquad\qquad 2x+2 & \\
\qquad\qquad 2x+2 & \\
\cline{1-1}
\end{array}$$

Trans. and completing the square, $x^2-2x+1=-2+1=-1$

Extract. the square root, $x-1=\pm\sqrt{-1}$, or $x=1\pm\sqrt{-1}$

Therefore the required roots are $1+\sqrt{-1}$ and $1-\sqrt{-1}$.

Ex. 3. Two roots of the equation $x^4-2x^3+2x^2+2x-3=0$ are 1 and $-1$. What are the other roots?

Divide the equation by $(x-1)\times(x+1)$ and the quotient will contain the required roots.

$$\begin{array}{l|l}
x^4-2x^3+2x^2+2x-3 & x^2-1 \\
\cline{2-2}
x^4 \qquad -\ x^2 & x^2-2x+3=0 \\
\cline{1-1}
\quad -2x^3+3x^2+2x & \\
\quad -2x^3 \qquad +2x & \\
\cline{1-1}
\qquad\qquad 3x^2-3 & \\
\qquad\qquad 3x^2-3 & \\
\cline{1-1}
\end{array}$$

Trans. and comp. the square, $x^2-2x+1=-3+1=-2$

Extract the square root, $x-1=\pm\sqrt{-2}$, or $x=1\pm\sqrt{-2}$.

Therefore the required roots are $1+\sqrt{-2}$ and $1-\sqrt{-2}$.

Ex. 4. Find the cubic equation of which two of the roots are $-2$ and $2-\sqrt{-1}$.

The remaining root is $2+\sqrt{-1}$, and the required equation is equal to the product of the factors

$$(x-2+\sqrt{-1}),\ (x-2-\sqrt{-1}),\ (x+2).$$

$$\begin{array}{l}
x-2+\sqrt{-1} \\
\underline{x-2-\sqrt{-1}} \\
x^2-2x+x\sqrt{-1} \\
\quad -2x \qquad\qquad -2\sqrt{-1}+4 \\
\underline{\qquad\qquad -x\sqrt{-1}+2\sqrt{-1}+1} \\
x^2-4x+5 \\
\underline{\qquad x+2} \\
x^3-4x^2+5x \\
\underline{\quad +2x^2-8x+10} \\
x^3-2x^2-3x+10=0. \quad \textit{Ans.}
\end{array}$$

ART. 560. Ex. 3. The equation $x^4-5x^2+4=0$ has four real roots. How many of them are positive?

The equation cannot have more than two positive roots, since it has only two variations. And as the whole number of real roots is four, there cannot be less than two negative roots. If now we supply the places of the deficient terms with $\pm 0$, the equation becomes

$$x^4\pm0-5x^2\pm0+4=0.$$

Here in either case the equation has two permanences, and hence cannot have more than two negative roots. Therefore it has two positive roots.

Ex. 4. The equation $x^5-x^4-5x^2-x+6=0$ has three real roots. How many of them are positive?

The equation cannot have more than two positive roots, since it has only two variations. And as the whole number of real roots is three, there must be at least one negative root. If now we supply the place of the absent term with $+0$, the equation becomes

$$x^5-x^4+0-5x^2-x+6=0;$$

in which there is only one permanence. And hence it appears that there cannot be more than one negative root. Therefore there are two positive roots.

ART. 562. Ex. 1. What are the equal roots of the equation

$$x^5+3x^4-x^3-7x^2+4=0 \qquad (1)$$

The derived polynomial of the first member is

$$5x^4+12x^3-3x^2-14x \qquad (2)$$

To find the greatest common divisor of (1) and (2), multiply (1) by 5, and divide the product by (2).

$$\begin{array}{l|l}
5x^5+15x^4-5x^3-35x^2+20 & \underline{5x^4+12x^3-3x^2-14x} \\
\underline{5x^5+12x^4-3x^3-14x^2} & \quad x \\
\qquad 3x^4-2x^3-21x^2+20=\text{1st Rem.} &
\end{array}$$

Multiply (2) by 3 and divide by the 1st Rem.

$$\begin{array}{l|l}
15x^4+36x^3-\phantom{0}9x^2-42x & \underline{3x^4-2x^3-21x^2+20} \\
\underline{15x^4-10x^3-105x^2\qquad+100} & \quad 5 \\
\qquad 46x^3+\phantom{0}96x^2-42x-100=\text{2d Rem.} &
\end{array}$$

Divide this by 2, and multiply the 1st Rem. by 23.

$$\begin{array}{l|l}
69x^4-\phantom{0}46x^3-483x^2+460 & \underline{23x^3+48x^2-21x-50} \\
\underline{69x^4+144x^3-\phantom{0}63x^2-150x} & \quad 3x \\
\qquad -190x^3-420x^2+150x+460=\text{3d Rem.} &
\end{array}$$

Change the signs of this remainder and divide by 10, and multiply the last divisor by 19.

$$\begin{array}{l|l}
437x^3+912x^2-399x-\phantom{0}950 & \underline{19x^3+42x^2-15x-46} \\
\underline{437x^3+966x^2-345x-1058} & \quad 23 \\
\qquad -\phantom{0}54x^2-\phantom{0}54x+\phantom{0}108=\text{4th Rem.} &
\end{array}$$

Change the signs of this remainder and divide by 54.

$$\begin{array}{l|l} 19x^3+42x^2-15x+46 & x^2+x-2 \\ \underline{19x^3+19x^2-38x} & 19x+23 \\ \quad 23x^2+23x-46 & \\ \quad \underline{23x^2+23x-46} & \end{array}$$

Therefore $(x^2+x-2)$ is the greatest common divisor of (1) and (2).

In order to solve the equation $x^2+x-2=0$,

Trans. and completing the square, $x^2+x+\frac{1}{4}=2\frac{1}{4}=\frac{9}{4}$

Extracting the square root, $x+\frac{1}{2}=\pm\frac{3}{2}$, and $x=1$ or $-2$.

Therefore $x^2+x-2=(x-1)\ (x+2)$; and hence, by the preceding theorem, the original equation contains the factors $(x-1)$, $(x+2)$, each raised to the second power: that is, the equation has two roots equal to 1, and two others equal to $-2$.

Divide the first member of the given equation by $(x-1)^2\times(x+2)^2$.

$$\begin{array}{l|l} x^5+3x^4-\ x^3-7x^2+4 & x^4+2x^3-3x^2-4x+4=(x-1)^2\times(x+2)^2. \\ \underline{x^5+2x^4-3x^3-4x^2+4x} & x+1 \\ \quad x^4+2x^3-3x^2-4x+4 & \\ \quad \underline{x^4+2x^3-3x^2-4x+4} & \end{array}$$

Therefore, by Art. 538, $-1$ is the fifth root of the equation.

Ex. 2. Find the equal and unequal roots of the equation

$$x^5+4x^4-14x^3-17x-6=0. \qquad (1)$$

The derived polynomial of the first member is

$$5x^4+16x^3-28x-17 \qquad (2)$$

To find the greatest common divisor of (1) and (2), multiply (1) by 5, and divide by (2).

$$\begin{array}{l|l} 5x^5+20x^4-70x^2-85x-30 & 5x^4+16x^3-28x-17 \\ \underline{5x^5+16x^4-28x^2-17x} & x \\ \quad 4x^4-42x^2-68x-30=\text{1st Rem.} & \end{array}$$

Divide this by 2, and multiply (2) by 2.

$$\begin{array}{l|l} 10x^4+\ 32x^3-\ 56x-\ 34 & 2x^4-21x^2-34x-15 \\ \underline{10x^4-105x^2-170x-\ 75} & 5 \\ \quad 32x^3+105x^2+114x+41=\text{2d Rem.} & \end{array}$$

Multiply the last divisor by 16, and divide by 2d Rem.

$$\begin{array}{l|l}
32x^4-336x^2-544x-240 & \underline{32x^3+105x^2+114x+41} \\
\underline{32x^4+105x^3+114x^2+41x} & \quad x \\
\quad -105x^3-450x^2-585x-240=3\text{d Rem.} &
\end{array}$$

Divide this by $-15$, and multiply the last divisor by 7.

$$\begin{array}{l|l}
224x^3+735x^2+798x+287 & \underline{7x^3+30x^2+39x+16} \\
\underline{224x^3+960x^2+1248x+512} & \quad 32 \\
\quad -225x^2-450x-225=4\text{th Rem.} &
\end{array}$$

Divide this by $-225$.

$$\begin{array}{l|l}
7x^3+30x^2+39x+16 & \underline{x^2+2x+1} \\
\underline{7x^3+14x^2+7x\quad} & 7x+16 \\
\quad 16x^2+32x+16 & \\
\quad \underline{16x^2+32x+16} &
\end{array}$$

Therefore $(x^2+2x+1)$ is the greatest common divisor of (1) and (2).

It is evident that $x^2+2x+1=(x+1)\times(x+1)$; and hence, by the preceding theorem, the original equation contains the factor $(x+1)$ raised to the third power: that is, the equation has three roots, each equal to $-1$.

Divide the first member of the given equation by $(x+1)^3$.

$$\begin{array}{l|l}
x^5+4x^4-14x^2-17x-6 & \underline{x^3+3x^2+3x+1}=(x+1)^3 \\
\underline{x^5+3x^4+3x^3+x^2\quad} & \quad x^2+x-6 \\
\quad x^4-3x^3-15x^2-17x & \\
\quad \underline{x^4+3x^3+3x^2+x\quad} & \\
\qquad -6x^3-18x^2-18x-6 & \\
\qquad \underline{-6x^3-18x^2-18x-6} &
\end{array}$$

The equation $x^2+x-6=0$ contains the remaining roots.

Trans. and completing the square, $x^2+x+\frac{1}{4}=6+\frac{1}{4}=\frac{25}{4}$

Extracting the square root, $x+\frac{1}{2}=\pm\frac{5}{2}$, and $x=2$ or $-3$

Therefore the roots of the given equation are $-1$, $-1$, $-1$, $2$, $-3$.

Ex. 3. Find the roots of the equation

$$x^5+x^4-14x^3+26x^2-19x+5=0. \qquad (1)$$

The derived polynomial of the first member is

$$5x^4+4x^3-42x^2+52x-19 \qquad (2)$$

To find the greatest common divisor of (1) and (2), multiply (1) by 5, and divide by (2).

$$\begin{array}{r|l}
5x^5+5x^4-70x^3+130x^2-95x+25 & 5x^4+4x^3-42x^2+52x-19 \\ \cline{2-2}
5x^5+4x^4-42x^3+\ 52x^2-19x & \qquad x \\ \cline{1-1}
x^4-28x^3+\ 78x^2-76x+25=\text{1st Rem.} &
\end{array}$$

$$\begin{array}{r|l}
5x^4+\ \ 4x^3-\ 42x^2+\ 52x-\ 19 & x^4-28x^3+78x^2-76x+25 \\ \cline{2-2}
5x^4-140x^3+390x^2-380x+125 & \qquad 5 \\ \cline{1-1}
144x^3-432x^2+432x-144=\text{2d Rem.} &
\end{array}$$

Divide this by 144.

$$\begin{array}{r|l}
x^4-28x^3+78x^2-76x+25 & x^3-3x^2+3x-1 \\ \cline{2-2}
x^4-\ 3x^3+\ 3x^2-\ \ x\qquad & \quad x-25 \\ \cline{1-1}
-25x^3+75x^2-75x+25 & \\
-25x^3+75x^2-75x+25 & \\ \cline{1-1}
\end{array}$$

Therefore $(x^3-3x^2+3x-1)$ is the greatest common measure of (1) and (2).

Now it is obvious that $(x^3-3x^2+3x-1)$ is equal to $(x-1)^3$; and hence by the preceding theorem the original equation contains the factor $(x-1)$ raised to the fourth power; that is, the equation has four roots each equal to 1.

Divide the first member of the equation by $(x-1)^4$.

$$\begin{array}{r|l}
x^5+\ x^4-14x^3+26x^2-19x+5 & x^4-4x^3+6x^2-4x+1=(x-1)^4. \\ \cline{2-2}
x^5-4x^4+\ 6x^3-\ 4x^2+\ \ x\qquad & \quad x+5 \\ \cline{1-1}
5x^4-20x^3+30x^2-20x+5 & \\
5x^4-20x^3+30x^2-20x+5 & \\ \cline{1-1}
\end{array}$$

Therefore, by Art. 538, $-5$ is the fifth root of the equation.

Ex. 4. Find the roots of the equation

$$x^6+2x^5-12x^4-14x^3+47x^2+12x-36=0 \qquad (1)$$

The derived polynomial of the first member is

$$6x^5+10x^4-48x^3-42x^2+94x+12 \qquad (2)$$

To find the greatest common divisor of (1) and (2), multiply (1) by 3, and divide (2) by 2, and then proceed as the rule directs.

$$\begin{array}{l|l}
3x^6+6x^5-36x^4-42x^3+141x^2+36x-108 & \underline{3x^5+5x^4-24x^3-21x^2+47x+6} \\
\underline{3x^6+5x^5-24x^4-21x^3+\ 47x^2+\ 6x} & \quad x \\
\qquad x^5-12x^4-21x^3+\ 94x^2+30x-108=\text{1st Rem.} &
\end{array}$$

$$\begin{array}{l|l}
3x^5+\ 5x^4-24x^3-\ 21x^2+47x+\ 6 & \underline{x^5-12x^4-21x^3+94x^2+30x-108} \\
\underline{3x^5-36x^4-63x^3+282x^2+90x-324} & \quad 3 \\
\qquad 41x^4+39x^3-303x^2-43x+330=\text{2d Rem.} &
\end{array}$$

Multiply the last divisor by 41.

$$\begin{array}{l|l}
41x^5-492x^4-861x^3+3854x^2+1230x-4428 & \underline{41x^4+39x^3-303x^2-43x+330} \\
\underline{41x^5+\ 39x^4-303x^3-\ \ 43x^2+\ 330x} & \quad x \\
\quad -531x^4-558x^3+3897x^2+\ 900x-4428=\text{3d Rem.} &
\end{array}$$

Change the signs of this remainder, and multiply the last divisor by 531.

$$\begin{array}{l|l}
21771x^4+20709x^3-160893x^2-22833x+175230 & \underline{531x^4+558x^3-3897x^2-900x+4428} \\
\underline{21771x^4+22878x^3-159777x^2-36900x+181548} & \quad 41 \\
\quad -\ 2169x^3-\ \ 1116x^2+14067x-\ \ 6318=\text{4th Rem.} &
\end{array}$$

Divide this remainder by —9, and multiply the last divisor by 241.

$$\begin{array}{l|l}
127971x^4+134478x^3-939177x^2-216900x+1067148 & \underline{241x^3+124x^2-1563x+702} \\
\underline{127971x^4+\ 65844x^3-829953x^2+372762x} & \quad 531x \\
\qquad 68634x^3-109224x^2-589662x+1067148=\text{5th Rem.} &
\end{array}$$

Divide this remainder by 18, and multiply the last divisor by 3813.

$$\begin{array}{l|l}
918933x^3+472812x^2-5959719x+2676726 & 3813x^3-6068x^2-32759x+59286 \\
\underline{918933x^3-1462388x^2-7894919x+14287926} & \quad 241 \\
\qquad 1935200x^2+1935200x-11611200=\text{6th Rem.} &
\end{array}$$

Divide this remainder by 1935200.

$$\begin{array}{l|l}
3813x^3-6068x^2-32759x+59286 & x^2+x-6 \\
\underline{3813x^3+3813x^2-22878x} & 3813x-9881 \\
\qquad -9881x^2-9881x+59286 & \\
\qquad \underline{-9881x^2-9881x+59286} &
\end{array}$$

Therefore $x^2+x-6$ is the common divisor sought. To solve the equation $x^2+x-6=0$,

Transposing and completing the square, $x^2+x+\frac{1}{4}=\frac{1}{4}+6=\frac{25}{4}$

Extracting the square root, $x+\frac{1}{2}=\pm\frac{5}{2}$, and $x=2$ or $-3$

Therefore $x^2+x-6=(x-2)\times(x+3)$; and hence the original equation contains the factors $(x-2)$, $(x+3)$, each raised to the second power: that is, the equation has two roots equal to 2, and two others equal to $-3$.

Divide the first member of the given equation by $(x-2)^2\times(x+3)^2$.

$$\begin{array}{l|l}
x^6+2x^5-12x^4-14x^3+47x^2+12x-36 & x^4+2x^3-11x^2-12x+36=(x-2)^2\times(x+3)^2 \\
\underline{x^6+2x^5-11x^4-12x^3+36x^2} & x^2-1 \\
\qquad -x^4-2x^3+11x^2+12x-36 & \\
\qquad \underline{-x^4-2x^3+11x^2+12x-36} &
\end{array}$$

Now $x^2-1=(x+1)\times(x-1)$; and therefore, by Art. 538, 1 and $-1$ are the remaining roots of the equation.

NOTE.—In the foregoing examples the process for obtaining the greatest common measure might have been different. The division is not carried as far as possible, but the remainder after one term of each quotient is obtained is in each case made a new divisor. This is done, not to shorten the solutions, but for convenience of notation, and greater uniformity in the work.

## Sturm's Theorem.

ART. 566. Ex. 2. How many real roots has the equation

$$2x^3+2x^2-8x-7=0\,?$$

Let the first member be denoted by $X$, and its derived polynomial by $X'$: then $X'=6x^2+4x-8$.

To find $X''$, multiply $X$ by 3, and divide the quotient by $X'$.

$$\begin{array}{r|l}
6x^3+6x^2-24x-21 & \underline{6x^2+4x-8} \\
\underline{6x^3+4x^2-\ \ 8x\qquad} & x+1 \\
2x^2-16x-21 &
\end{array}$$

Multiply the remainder by 3, and divide again.

$$\begin{array}{r}
6x^2-48x-63 \\
\underline{6x^2+\ \ 4x-\ \ 8} \\
-52x-55
\end{array}\qquad \text{Therefore } X''=52x+55.$$

Divide $X'$ by 2, multiply the quotient by 52, and divide the product by $X''$.

$$\begin{array}{r|l}
156x^2+104x-208 & \underline{52x+55} \\
\underline{156x^2+165x\qquad\quad} & 3x-61 \\
-\ 61x-208 &
\end{array}$$

Multiply the remainder by 52, and divide again.

$$\begin{array}{r}
-3172x-10816 \\
\underline{-3172x-\ \ 3355} \\
-\ \ 7461
\end{array}\qquad \text{Therefore } X'''=+7461.$$

The remainder of the solution is given in the Algebra.

Ex. 3. How many real roots has the equation

$$x^4+4x^3+x^2-16x-18=0\,?$$

Let the first member be denoted by $X$, and its derived polynomial by $X'$: then $X'=4x^3+12x^2+2x-16$.

To find $X''$, multiply $X$ by 2, divide $X'$ by 2, and divide the product by the quotient.

$$\begin{array}{r|l} 2x^4+8x^3+2x^2-32x-36 & 2x^3+6x^2+x-8 \\ \cline{2-2} \underline{2x^4+6x^3+x^2-8x\phantom{-36}} & \quad x+1 \\ 2x^3+x^2-24x-36 & \\ \underline{2x^3+6x^2+x-8} & \\ -5x^2-25x-28 & \end{array}$$

Therefore $X''=5x^2+25x+28$.

To find $X'''$, multiply the last divisor by 5, and divide by $X''$.

$$\begin{array}{r|l} 10x^3+30x^2+5x-40 & 5x^2+25x+28 \\ \cline{2-2} \underline{10x^3+50x^2+56x\phantom{-40}} & \quad 2x-4 \\ -20x^2-51x-40 & \\ \underline{-20x^2-100x-112} & \\ 49x+72 & \end{array}$$

Therefore $X'''=-49x-72$.

To find $X''''$, multiply $X''$ by 49, and divide by $X'''$.

$$\begin{array}{r|l} 245x^2+1225x+1372 & -49x-72 \\ \cline{2-2} \underline{245x^2+360x\phantom{+1372}} & \quad -5x-865 \\ 865x+1372 & \end{array}$$

Multiply this remainder by 49, and carry out the division.

$$\begin{array}{r} 42385x+67228 \\ \underline{42385x+62280} \\ 4948 \end{array}$$

Therefore $X''''=-4948$.

The remainder of the solution is given in the Algebra.

Ex. 4. How many real roots has the equation

$$x^3+3x^2+7x+4=0\ ?$$

Let the first member by denoted by $X$, and its derived polynomial by $X'$: then $X'=3x^2+6x+7$.

To find $X''$, multiply $X$ by 3, and divide by $X'$.

$$\begin{array}{r|l} 3x^3+9x^2+21x+12 & 3x^2+6x+7 \\ \cline{2-2} \underline{3x^3+6x^2+7x\phantom{+12}} & \quad x+1 \\ 3x^2+14x+12 & \\ \underline{3x^2+6x+7} & \\ 8x+5 & \end{array}$$

Therefore $X''=-8x-5$.

To find $X'''$, multiply $X'$ by 8, and divide by $X''$.

$$\begin{array}{r|l}
24x^2+48x+56 & \underline{-8x-5} \\
\underline{24x^2+15x} & -3x-33 \\
33x+56 &
\end{array}$$

Multiply this remainder by 8, and divide again.

$$\begin{array}{r}
264x+448 \\
\underline{264x+165} \\
283
\end{array}\qquad \text{Therefore } X'''=-283.$$

Collecting the values of $X$, $X'$, &c.

$$X=x^3+3x^2+7x+4, \qquad X''=-8x-5,$$
$$X'=3x^2+6x+7 \qquad X'''=-283.$$

When $x=-\infty$ the series of signs is $-++-$ giving 2 variations.
" $x=\infty$ " " $++--$ " 1 "
" $x=0$ " " $++--$ " 1 "
" $x=-1$ " " $-++-$ " 2 "

Therefore the equation has only 1 real root, and its value is between 0 and $-1$.

Ex. 5. How many real roots has the equation

$$x^4-4x^2-8x-4=0\,?$$

In this problem $X'=4x^3-8x-8$.

To find $X''$, divide $X'$ by 4, and divide $X$ by the quotient.

$$\begin{array}{r|l}
x^4-4x^2-8x-4 & \underline{x^3-2x-2} \\
\underline{x^4-2x^2-2x} & x \\
-2x^2-6x-4 &
\end{array}\qquad \text{Therefore } X''=2x^2+6x+4.$$

To find $X'''$, divide $X'$ by 4, and $X''$ by 2, and then divide the first quotient by the second.

$$\begin{array}{r|l}
x^3-2x-2 & \underline{x^2+3x+2} \\
\underline{x^3+3x^2+2x} & x-3 \\
-3x^2-4x-2 & \\
\underline{-3x^2-9x-6} & \\
5x+4 &
\end{array}\qquad \text{Therefore } X'''=-5x-4.$$

To find $X''''$, divide $X''$ by 2, multiply it by 5, and then divide by $X'''$.

$$\begin{array}{r|l} 5x^2+15x+10 & -5x-4 \\ \cline{2-2} \underline{5x^2+\ \ 4x\phantom{+10}} & -x-11 \\ 11x+10 & \end{array}$$

Multiply this remainder by 5, and divide again.

$$\begin{array}{r} 55x+50 \\ \underline{55x+44} \\ 6 \end{array}$$

Therefore $X''''=-6$.

Collecting the values of $X$, $X'$, &c.

$$\begin{array}{ll} X & =x^4-4x^2-8x-4 \\ X' & =x^3-2x-2=(4x^3-8x-8)\div 4 \\ X'' & =x^2+3x+2=(2x^2+6x+4)\div 2 \\ X''' & =-5x-4 \\ X'''' & =-6 \end{array}$$

When $x=-\infty$ the series of signs is $+-++-$ giving 3 varia's.

" $x=\ \infty$ " " $+++--$ " 1 "

" $x=\ 0$ " " $--+--$ " 2 "

" $x=\ 1$ " " $--+--$ " 2 "

" $x=\ 2$ " " $-++--$ " 2 "

" $x=\ 3$ " " $+++--$ " 1 "

" $x=-1$ " " $+-\pm+-$ " 3 "

Therefore the equation has two real roots; a positive one between 2 and 3, and a negative one between 0 and $-1$.

Ex. 6. How many real roots has the equation

$$2x^3-5x-8=0?$$

In this problem $X'=6x^2-5$.

To find $X''$, multiply $X$ by 3, and divide by $X'$.

$$\begin{array}{r|l} 6x^3-15x-24 & 6x^2-5 \\ \cline{2-2} \underline{6x^3-\ \ 5x\phantom{-24}} & x \\ -10x-24 & \end{array}$$

Divide this remainder by 2; then $X''=5x+12$.

To find $X'''$, multiply $X'$ by 5, and divide by $X''$.

$$\begin{array}{r|l} 30x^2-25\phantom{x} & 5x+12 \\ \cline{2-2} \underline{30x^2+72x\phantom{-25}} & 6x-72 \\ -72x-25 & \end{array}$$

Multiply this remainder by 5, and divide again.

$$\begin{array}{r} -360x-125 \\ \underline{-360x-864} \\ 739 \end{array} \qquad \text{Therefore } X'''=-739.$$

Collecting the values of $X$, $X'$, &c.

$$X\ =2x^3-5x-8, \qquad X''\ =5x+12,$$
$$X'=6x^2-5, \qquad X'''=-739.$$

When $x=-\infty$ the series of signs is $-+--$ giving 2 variations.
" $x=\ \ \infty$ " " $+++-$ " 1 "
" $x=\ \ 0$ " " $--+-$ " 2 "
" $x=\ \ 1$ " " $-++-$ " 2 "
" $x=\ \ 2$ " " $-++-$ " 2 "
" $x=\ \ 3$ " " $+++-$ " 1 "

Therefore the equation has only one real root, and its value is between 2 and 3.

Ex. 7. How many real roots has the equation

$$x^4-7x^2-2x+2=0\,?$$

In this problem $X'=2x^3-7x-1=(4x^3-14x-2)\div 2.$

To find $X''$, multiply $X$ by 2, and divide by $X'$.

$$\begin{array}{r|l} 2x^4-14x^2-4x+4 & 2x^3-7x-1 \\ \cline{2-2} \underline{2x^4-\ \ 7x^2-\ \ x} & \quad x \\ -\ \ 7x^3-3x+4 & \end{array} \qquad \text{Therefore } X''=7x^2+3x-4.$$

To find $X'''$, multiply $X'$ by 7, and divide by $X''$.

$$\begin{array}{r|l} 14x^3-49x-\ \ 7 & 7x^2+3x-4 \\ \cline{2-2} \underline{14x^3+\ \ 6x^2-\ \ 8x} & \quad 2x-6 \\ -\ \ 6x^3-41x-7 & \end{array}$$

Multiply this remainder by 7, and divide again.

$$\begin{array}{r} -42x^2-287x-49 \\ \underline{-42x^2-\ \ 18x+24} \\ -269x-73 \end{array} \qquad \text{Therefore } X'''=269x+73.$$

To find $X''''$, multiply $X''$ by 269, and divide by $X'''$.

$$\begin{array}{r|l} 1883x^2+807x-1076 & 269x+73 \\ \cline{2-2} \underline{1883x^2+511x\qquad\quad} & \quad 7x+74 \\ 296x-1076 & \end{array}$$

Divide this remainder by 4, multiply it by 269, and carry out the division.

$$\begin{array}{r} 19906x-72361 \\ \underline{19906x+\ \ 5402} \\ -77763 \end{array}\qquad \text{Therefore } X''''=77763.$$

Collecting the values of $X$, $X'$, &c.

$$\begin{array}{ll} X\ =x^4-7x^2-2x+2, & X''\ =7x^2+3x-4, \\ X'=2x^3-7x-1, & X'''=269x+73, \end{array}$$

$$X''''=77763.$$

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| When | $x=-\infty$ | the series of | signs is | $+-+-+$ | giving | 4 | varia's. |
| " | $x=\ \ \infty$ | " | " | $+++++$ | " | 0 | " |
| " | $x=\ \ 0$ | " | " | $+--++$ | " | 2 | " |
| " | $x=\ \ 1$ | " | " | $--+++$ | " | 1 | " |
| " | $x=\ \ 2$ | " | " | $-++++$ | " | 1 | " |
| " | $x=\ \ 3$ | " | " | $+++++$ | " | 0 | " |
| " | $x=-1$ | " | " | $-+\pm-+$ | " | 3 | " |
| " | $x=-2$ | " | " | $--+-+$ | " | 3 | " |
| " | $x=-3$ | " | " | $+-+-+$ | " | 4 | " |

Therefore the equation has four real roots; of which one is between 0 and 1, another between 2 and 3, a third between 0 and −1, and a fourth between −2 and −3.

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## Elimination.

Art. 567. Ex. 2. Eliminate $x$ from the equations

$$x^2+y-1=0 \qquad (1)$$

$$2xy-y-1=0 \qquad (2)$$

Multiply (1) by $2y$, and divide by (2).

$$\begin{array}{l|l} 2yx^2+2y^2-2y & \underline{2yx-y-1} \\ \underline{2yx^2-(y+1)x\qquad} & x+(y+1) \\ \quad (y+1)x+2y^2-2y & \end{array}$$

Multiply this remainder by $2y$, and divide again.

$$\begin{array}{r} 2xy(y+1)+4y^3-4y^2 \\ \underline{2xy(y+1)-(y+1)^2} \\ (y+1)^2+4y^3-4y^2 \end{array}$$

Now this remainder does not contain $x$, and therefore uniting terms, we have,

$$4y^3-3y^2+2y+1=0. \quad \textit{Ans.}$$

Ex. 3. Eliminate $x$ from the equations

$$2x^2-xy^2+y^3-1=0 \qquad (1)$$
$$x^2-xy+y^2-1=0 \qquad (2)$$

$$\begin{array}{l|l} 2x^2-\ xy^2+\ y^3-1 & \underline{x^2-xy+y^2-1} \\ \underline{2x^2-2xy\ +2y^2-2} & \quad 2 \\ \quad (2y-y^2)x-2y^2+y^3+1=\text{1st Rem.} & \end{array}$$

Multiply the last divisor by $(2y-y^2)$, and divide by 1st Rem.

$$\begin{array}{l|l} (2y-y^2)x^2-(2y^2-y^3)x+2y^3-y^4-2y+y^2 & \underline{(2y-y^2)x-(2y^2-y^3)+1} \\ \underline{(2y-y^2)x^2-(2y^2-y^3)x+x} & \quad x-1 \\ \qquad -x+2y^3-y^4-2y+y^2 & \end{array}$$

Multiply by $(2y-y^2)$, and divide again.

$$\begin{array}{l} -(2y-y^2)x+(2y-y^2)\,(2y^3-y^4-2y+y^2) \\ \underline{-(2y-y^2)x+(2y^2-y^3)-1} \\ \qquad (2y-y^2)\,(2y^3-y^4-2y+y^2)-(2y^2-y^3)+1=\text{2d Rem.} \end{array}$$

Now this remainder does not contain $x$; therefore expanding, we have,

$$y^6-4y^5+3y^4+5y^3-6y^2+1=0. \quad \textit{Ans.}$$

Ex. 4. Eliminate $x$ from the equations

$$x^3+y^3-a=0 \qquad (1)$$
$$x^2+xy+y^2-b=0 \qquad (2)$$

$$\begin{array}{l|l} x^3+y^3\ -a & \underline{x^2+xy+y^2-b} \\ \underline{x^3+x^2y+xy^2-bx} & \quad x-y \\ \quad -x^2y-xy^2+bx+\ y^3-a & \\ \quad \underline{-x^2y-xy^2\quad -\ y^3\quad +by} & \\ \qquad\qquad bx+2y^3-by-a=\text{1st Rem.} & \end{array}$$

Multiply the last divisor by $b$, and divide by 1st Rem.

$$\begin{array}{l|l} bx^2+bxy+by^2-b^2 & \underline{bx+2y^3-by-a} \\ \underline{bx^2-bxy+2xy^3-ax} & \quad x+(2by-2y^3+a) \\ \quad (2by-2y^3+a)x+by^2-b^2 & \end{array}$$

Multiply this remainder by $b$, and divide again.

$$\begin{array}{r}
b(2by-2y^3+a)x+b^2y^2-b^3 \\
\underline{b(2by-2y^3+a)x+(2by-2y^3+a)\ (2y^3-by-a)} \\
-(2by-2y^3+a)\ (2y^3-by-a)+b^2y^2-b^3
\end{array}$$

Expanding and arranging according to the powers of $y$, we obtain

$$4y^6-6by^4-4ay^3+3b^2y^2+3aby+a^2-b^3=0. \quad \textit{Ans.}$$

Ex. 5. Eliminate $x$ from the equations

$$x^2+ay-b=0 \qquad (1)$$
$$y^2+cx-d=0 \qquad (2)$$

Multiply (1) by $c$, and divide by (2).

$$\begin{array}{l|l}
cx^2+acy-bc & cx+y^2-d \\
\cline{2-2}
\underline{cx^2+xy^2-dx} & x+(d-y^2) \\
(d-y^2)x+acy-bc &
\end{array}$$

Multiply this remainder by $c$, and divide again.

$$\begin{array}{r}
(cd-cy^2)x+ac^2y-bc^2 \\
\underline{(cd-cy^2)x+dy^2-y^4-d^2+dy^2} \\
-dy^2+y^4+d^2-dy^2+ac^2y-bc^2
\end{array}$$

And uniting terms, $\quad y^4-2dy^2+ac^2y+d^2-bc^2=0. \quad$ *Ans.*

Ex. 6. Eliminate $x$ and $y$ from the equations

$$xy-z-10=0 \qquad (1)$$
$$xz-y+11=0 \qquad (2)$$
$$yz-x+14=0 \qquad (3)$$

Multiplying (1) by $z$, $\qquad xyz-z^2-10z=0 \quad (4)$
" (2) " $y$, $\qquad xyz-y^2+11y=0 \quad (5)$
Subtracting (5) from (4), $\qquad y^2-z^2-10z-11y=0 \quad (6)$
Multiplying (3) by $z$, $\qquad yz^2-xz+14z=0 \quad (7)$
Adding (2) and (7), $\qquad yz^2-y+14z+11=0 \quad (8)$

To eliminate $y$ from (6) and (8), multiply (6) by $(z^2-1)$, and divide the product by (8).

$$\begin{array}{l|l}
y(y-11)\ (z^2-1)-(z^2+10z)\ (z^2-1) & y(z^2-1)+14z+11 \\
\cline{2-2}
\underline{y(y-11)\ (z^2-1)+(14z+11)\ (y-11)} & (y-11)+(14z+11) \\
-(14z+11)\ (y-11)-(z^2+10z)\ (z^2-1) &
\end{array}$$

Multiply this remainder by $-(z^2-1)$, and divide again.

$$\begin{array}{r}
y(14z+11)\,(z^2-1)-11(14z+11)\,(z^2-1)+(z^2+10z)\,(z^2-1)^2 \\
\underline{y(14z+11)\,(z^2-1)+\quad(14z+11)^2\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad} \\
-(14z+11)^2-11(14z+11)\,(z^2-1)+(z^2+10z)\,(z^2-1)^2
\end{array}$$

Expanding, uniting terms, and dividing by $z$, we obtain

$$z^5+10z^4-2z^3-174z^2-316z-144=0. \quad \textit{Ans.}$$

Ex. 7. Eliminate $x$ and $y$ from the equations

$$x+xy-a=0 \qquad (1)$$
$$y+yz-b=0 \qquad (2)$$
$$z+zx-c=0 \qquad (3)$$

Multiply (1) by $z$, and divide the product by (3).

$$\begin{array}{l|l}
xz(1+y)-az & \underline{xz+z-c} \\
\underline{xz(1+y)+(1+y)\,(z-c)} & \quad (1+y) \\
\quad -(1+y)\,(z-c)-az &
\end{array}$$

Therefore a new equation which does not contain $x$ is

$$cy-zy-z+c-az=0 \qquad (4)$$

To eliminate $y$, combine (4) with (2) by multiplying (4) by $(1+z)$, and dividing by (2).

$$\begin{array}{l|l}
y(c-z)\,(1+z)+(c-z)\,(1+z)-az(1+z) & \underline{y(1+z)-b} \\
\underline{y(c-z)\,(1+z)-b(c-z)\qquad\qquad\qquad\quad} & \quad (c-z) \\
\quad b(c-z)+(c-z)\,(1+z)-az(1+z) &
\end{array}$$

Or, $\qquad bc-bz+c+cz-z-z^2-az-az^2$

And collecting the coefficients of the several powers of $z$, and multiplying by $-1$,

$$(a+1)z^2+(a+b-c+1)z-c(b+1)=0. \quad \textit{Ans.}$$

Ex. 8. Eliminate $x$ and $y$ from the equations

$$y(x+y-z)=a \qquad (1)$$
$$z(y+z-x)=b \qquad (2)$$
$$x(z+x-y)=c \qquad (3)$$

Multiplying (2) by $x$, $\qquad xyz+xz^2-x^2z-bx=0 \qquad (4)$

" (3) " $z$, $\qquad -xyz+xz^2+x^2z-cz=0 \qquad (5)$

Adding (4) and (5), $\qquad 2xz^2-bx-cz=0 \qquad (6)$

Multiply (1) by $z$, and divide by (2).

$$yz(x+y-z)-az \quad \Big| \; zy+z^2-xz-b$$

$$\underline{yz(x+y-z)+(z^2-xz-b)\,(x+y-z)} \qquad (x+y-z)-(z^2-xz-b)$$

$$-y(z^2-xz-b)-(x-z)\,(z^2-xz-b)-az$$

Multiply this remainder by $z$, and divide again.

$$-yz(z^2-xz-b)-(xz-z^2)\,(z^2-xz-b)-az^2$$

$$\underline{-yz(z^2-xz-b)-(z^2-xz-b)^2}$$

$$(z^2-xz-b)^2-(xz-z^2)\,(z-xz-b)-az^2$$

Or, $\quad 2z^4-4xz^3+2x^2z^2-3bz^2+3bxz+b^2-az^2$

Multiply this remainder by $(2z^2-b)$, and divide by (6).

$$x(2xz^2+3bz-4z^3)\,(2z^2-b)+(2z^4-3bz^2-az^2+b^2)\,(2z^2-b) \quad \Big| \; x(2z^2-b)-cz$$

$$\underline{x(2xz^2+3bz-4z^3)\,(2z^2-b)-cz(2xz^2+3bz-4z^3)} \qquad (2xz^2+3bz-4z^3)+2cz^3$$

$$cz(2xz^2+3bz-4z^3)+(2z^4-3bz^2-az^2+b^2)\,(2z^2-b)$$

Multiply this remainder by $(2z^2-b)$, and divide again.

$$x(2cz^3)\,(2z^2-b)+cz(3bz-4z^3)\,(2z^2-b)+(2z^4-3bz^2-az^2+b^2)\,(2z^2-b)^2$$

$$\underline{x(2cz^3)\,(2z^2-b)-cz(2cz^3)}$$

$$cz(2cz^3)+cz(3bz-4z^3)\,(2z^2-b)+(2z^4-3bz^2-az^2+b^2)\,(2z^2-b)^2$$

Or, $\quad 2c^2z^4+10bcz^4-8cz^6-3b^2cz^2+8z^8-20bz^6-4az^6+4abz^4+18b^2z^4-ab^2z^2-7b^3z^2+b^4.$

Collecting the coefficients of the several powers of $x$,

$$8z^8-4(a+5b+2c)z^6+2(2ab+9b^2+5bc+c^2)z^4-b^2(a+7b+3c)z^2+b^4=0.$$

## RESOLUTION OF EQUATIONS.

ART. 570. Ex. 3. Find the roots of the equation

$$x^4-30x^3+869x-30=0.$$

Here the divisors of the last term are 1, $-1$, 2, $-2$, 3, $-3$, 5, $-5$, &c. Upon trial it is found that none of these fulfil the equations of condition, except $-5$. For this number the said equations become

$$\frac{-30}{-5}=6=C' \qquad \frac{0-175}{-5}=35=A'$$

$$\frac{869+6}{-5}=-175=B' \qquad \frac{-30+35}{-5}=-1$$

Therefore $-5$ is a root of the given equation, and the next lower equation is

$$x^3-35x^2+175x-6=0.$$

The divisors of the last term of this equation, except 6 and $-6$, have been already tried, and found not to be roots of the given equation. It remains, therefore, only to try 6 and $-6$. For 6, the equations of condition become

$$\frac{-6}{6}=-1=B' \quad \frac{175-1}{6}=29=A' \quad \frac{-35+29}{6}=-1$$

Therefore 6 is a root of the given equation, and the next lower equation is

$$x^2-29x+1=0.$$

Now, according to the Rule given on page 57, the values of $x$ in this equation are

$$x=\frac{29}{2}\pm\sqrt{-1+\left(\frac{29}{2}\right)^2}=\frac{29}{2}\pm\sqrt{\frac{-4+841}{4}}=\frac{29\pm\sqrt{837}}{2}$$

Therefore the roots of the given equation are

$$-5,\ 6,\ \tfrac{1}{2}(29\pm\sqrt{837}).$$

Ex. 4. Find the roots of the equation

$$x^5-2x^4-40x^3-10x^2+279x+252=0.$$

The divisors of the last term are 1, $-1$, 2, $-2$, 3, $-3$, 4, $-4$, 6, $-6$, &c. For $-1$, the equations of condition become

$$-2-1=-3 \qquad 279-27=252$$
$$-40+3=-37 \qquad 252-252=0$$
$$-10+37=27$$

Therefore $-1$ is a root of the equation, and the next lower equation is

$$x^4-3x^3-37x^2+27x+252=0.$$

Upon trial it is found that 1, 2, $-2$, are not roots of the equation. But if we try 3, we obtain

$$\frac{252}{3}=84 \quad \frac{27+84}{3}=37 \quad \frac{-37+37}{3}=0 \quad \frac{-3+0}{3}=-1$$

Therefore 3 is one of the required roots, and the next lower equation is

$$x^3-37x-84=0.$$

For $-3$ the equations of condition become

$$\frac{-84}{-3}=28 \qquad \frac{-37+28}{-3}=3 \qquad \frac{0+3}{-3}=-1$$

Therefore $-3$ is one of the required roots, and the next lower equation is

$$x^2-3x-28=0.$$

According to the Rule given on page 57,

$$x=\frac{3}{2}\pm\sqrt{28+\frac{9}{4}}=\frac{3}{2}\pm\sqrt{\frac{112+9}{4}}=\frac{3}{2}\pm\frac{11}{2}=7 \text{ or } -4$$

Therefore the required roots are $-1$, 3, $-3$, $-4$, 7.

Ex. 5. Find the roots of the equation

$$x^6-8x^5+6x^4+61x^3-60x^2-116x+80=0.$$

The divisors of the last term are 1, $-1$, 2, $-2$, 4, $-4$, &c. Of these 1 and $-1$ are found upon trial not to be roots. Trying 2, we obtain

$$\frac{80}{2}=40 \qquad \frac{-60-38}{2}=-49 \qquad \frac{6+6}{2}=6$$
$$\frac{-116+40}{2}=-38 \qquad \frac{61-49}{2}=6 \qquad \frac{-8+6}{2}=-1$$

Hence 2 is a root, and the next lower equation is

$$x^5-6x^4-6x^3+49x^2+38x-40=0.$$

Trying $-2$ by the preceding rule, we obtain,

$$\frac{-40}{-2}=20 \qquad \frac{49-29}{-2}=-10 \qquad \frac{-6+8}{-2}=-1$$

$$\frac{38+20}{-2}=-29 \qquad \frac{-6-10}{-2}=8$$

Hence $-2$ is a root, and the next lower equation is

$$x^4-8x^3+10x^2+29x-20=0.$$

The divisors 2 and $-2$ are found unsatisfactory, but 4 gives

$$\frac{-20}{4}=-5 \qquad \frac{29-5}{4}=6 \qquad \frac{10+6}{4}=4 \qquad \frac{-8+4}{4}=-1$$

Therefore 4 is a root, and the next lower equation is

$$x^3-4x^2-6x+5=0.$$

For the divisor 5, the successive quotients are

$$\frac{5}{5}=1 \qquad \frac{-6+1}{5}=-1 \qquad \frac{-4-1}{5}=-1$$

Hence 5 is a root, and the next equation is $x^2+x-1=0$.

According to the Rule given on page 57, we have

$$x=-\frac{1}{2}\pm\sqrt{1+\frac{1}{4}}=-\frac{1}{2}\pm\sqrt{\frac{5}{4}}=\frac{-1\pm\sqrt{5}}{2}$$

And the roots of the given equation are 2, $-2$, 4, 5, $\frac{-1\pm\sqrt{5}}{2}$.

Ex. 6. Find the roots of the equation

$$x^7-x^6-14x^5+14x^4+49x^3-49x^2-36x+36=0.$$

For the divisor 1 the equations of condition become

$$-1+1=0 \qquad 49+0=49 \qquad -36+0=-36$$

$$-14+0=-14 \qquad -49+49=0 \qquad 36-36=0$$

$$14-14=0$$

Therefore 1 is a root, and the next lower equation is

$$x^6-14x^4+49x^2-36=0.$$

Trying 1 again, we obtain

$$0+1=1 \qquad 0-13=-13 \qquad 0+36=36$$

$$-14+1=-13 \qquad 49-13=36 \qquad -36+36=0$$

Therefore a second root equals 1, and the next lower equation is

$$x^5+x^4-13x^3-13x^2+36x+36=0.$$

Here 1 being tried again is found unsatisfactory, but $-1$ gives the following,

$$1-1=0 \qquad -13+13=0 \qquad 36-36=0$$
$$-13-0=-13 \qquad 36-0=36$$

Therefore $-1$ is a root, and the next lower equation is

$$x^4-13x^2+36=0.$$

Finding the value of $x^2$ in this equation by the Rule given on page 57,

$$x^2=\frac{13}{2}\pm\sqrt{-36+\left(\frac{13}{2}\right)^2}=\frac{13}{2}\pm\sqrt{\frac{-144+169}{4}}=\frac{13\pm5}{2}=9 \text{ or } 4$$

Extracting the square root, $x=\pm3$, and $x=\pm2$.

And the required roots are 1, 1, $-1$, 3, $-3$, 2, $-2$.

Ex. 7. Find the roots of the equation

$$x^4+4x^3-36x^2-37x-40=0.$$

The divisors of the last term are 1, $-1$, 2, $-2$, 4, $-4$, 5, &c. Upon trying these successively they all fail to answer the conditions of a root, until 5 is reached, which gives

$$\frac{-40}{5}=-8 \qquad \frac{-37-8}{5}=-9 \qquad \frac{-36-9}{5}=-9 \qquad \frac{4-9}{5}=-1$$

Therefore 5 is a root, and the next lower equation is

$$x^3+9x^2+9x+8=0.$$

The divisors of 8 have been tried already, except 8 and $-8$. For $-8$ the equations of condition become

$$\frac{8}{-8}=-1 \qquad \frac{9-1}{-8}=-1 \qquad \frac{9-1}{-8}=-1$$

$-8$ is proved to be a root, and the next lower equation is

$$x^2+x+1=0.$$

By the Rule given on p. 57, $x=-\frac{1}{2}\pm\sqrt{-1+\frac{1}{4}}=\frac{-1\pm\sqrt{-3}}{2}$

And the roots of the given equation are

$$5,\ -8,\ \frac{-1+\sqrt{-3}}{2},\ \frac{-1-\sqrt{-3}}{2}.$$

Ex. 8. Find the roots of the equation

$$x^5-32x^3+6x^2+175x-150=0.$$

Trying the divisor 1, the equations of condition become

$$0+1=1 \qquad 6-31=-25 \qquad -150+150=0$$
$$-32+1=-31 \qquad 175-25=150$$

Therefore 1 is a root, and the next lower equation is

$$x^4+x^3-31x^2-25x+150=0$$

Here 1 and $-1$ when tried are found to fail. But 2 gives

$$\frac{150}{2}=75 \qquad \frac{-25+75}{2}=25 \qquad \frac{-31+25}{2}=-3 \qquad \frac{1-3}{2}=-1$$

Hence 2 also is a root, and the next lower equation is

$$x^3+3x^2-25x-75=0.$$

For the divisor $-3$ the equations of condition are

$$\frac{-75}{-3}=25 \qquad \frac{-25+25}{-3}=0 \qquad \frac{3+0}{-3}=-1$$

$-3$ is a root, and the next lower equation is $x^2-25=0$.

Transposing and extracting the square root, $x=\pm5$.

And the roots of the given equation are 1, 2, $-3$, 5, $-5$.

## Approximation by Double Position.

Art. 572. Ex. 2. What are the roots of the equation

$$x^3-8x^2+4x+48=0\,?$$

Let 4.1 and 4.2 be the assumed numbers.

$$\left.\begin{array}{ll}\text{Then} & (4.1)^3-8(4.1)^2+4(4.1)+48=\text{1st error} \\ \text{Or} & 68.921-134.48+16.4+48=-1.159\end{array}\right\}$$

$$\left.\begin{array}{ll}\text{Also} & (4.2)^3-8(4.2)^2+4(4.2)+48=\text{2d error} \\ \text{Or} & 74.088-141.12+16.8+48=-2.232\end{array}\right\}$$

Difference of errors $=-2.232-(-1.159)=-1.073$.

Therefore, $-1.073 : 0.1 :: -1.159 : 0.108$, the correction.

$4.1-.108=3.992$. Let $x=3.99$ or 3.98.

$$\left.\begin{array}{ll}\text{Then} & (3.99)^3-8(3.99)^2+4(3.99)+48=\text{1st error} \\ \text{Or} & 63.5212-127.3608+15.96+48=0.1204\end{array}\right\}$$

$$\left.\begin{array}{ll}\text{Also} & (3.98)^3-8(3.98)^2+4(3.98)+48=\text{2d error}\\ \text{Or} & 63.0448-126.7232+15.92+48=0.2416\end{array}\right\}$$

Difference of errors $=0.2416-0.1204=0.1212$.

Therefore, $0.1212 : 0.01 :: 0.1204 : .01$, the correction.

$3.99+.01=4$, which satisfies the equation and is a root.

To find the remaining roots, divide the first member of the given equation by $(x-4)$.

$$(x^3-8x^2+4x+48)\div(x-4)=(x^2-4x-12)$$

This quotient placed equal to zero is the equation which contains the required roots. But in this equation, by the Rule given on page 57,

$$x=2\pm\sqrt{+12+4}=2\pm4=6 \text{ or } -2.$$

Therefore the roots of the given equation are $-2$, 4, 6.

Ex. 3. What are the roots of the equation

$$x^3-16x^2+65x-50=0.$$

Let 5.1 and 5.2 be the assumed numbers.

$$\left.\begin{array}{ll}\text{Then} & (5.1)^3-16(5.1)^2+65(5.1)-50=\text{1st error}\\ \text{Or} & 132.651-416.16+331.5-50=-2.009\end{array}\right\}$$

$$\left.\begin{array}{ll}\text{Also} & (5.2)^3-16(5.2)^2+65(5.2)-50=\text{2d error}\\ \text{Or} & 140.608-432.64+338-50=-4.032\end{array}\right\}$$

Difference of errors $=-4.032-(-2.009)=-2.023$.

Therefore, $-2.023 : 0.1 :: -2.009 : 0.099$, the correction

$5.1-0.099=5.001$. Let $x=5.001$ or $5.002$.

$$\left.\begin{array}{ll}\text{Then} & (5.001)^3-16(5.001)^2+65(5.001)-50=\text{1st error}\\ \text{Or} & 125.075-400.16+325.065-50=-0.02\end{array}\right\}$$

$$\left.\begin{array}{ll}\text{Also} & (5.002)^3-16(5.002)^2+65(5.002)-50=\text{2d error}\\ \text{Or} & 125.15-400.32+325.13-50=-0.04\end{array}\right\}$$

Difference of errors $=-0.04-(-0.02)=-0.02$.

Therefore, $-0.02 : 0.001 :: -0.02 : 0.001$, the correction.

$5.001-0.001=5$, which satisfies the equation, and is a root.

$$(x^3-16x^2+65x-50)\div(x-5)=x^2-11x+10=0.$$

The last equation contains the remaining roots. Finding the value of $x$ in this equation by the Rule given on page 57, we have,

$$x=\frac{11}{2}\pm\sqrt{-10+\left(\frac{11}{2}\right)}=\frac{11}{2}\pm\sqrt{\frac{-40+121}{4}}=\frac{11}{2}\pm\frac{9}{2}=10 \text{ or } 1$$

And the roots of the given equation are 1, 5, 10.

Ex. 4. What are the roots of the equation

$$x^3+2x^2-33x=90\,?$$

Transpose 90, and suppose one of the roots to be 6.1 or 6.2.

$$\left.\begin{array}{ll}\text{Then} & (6.1)^3+2(6.1)^2-33(6.1)-90=\text{1st error}\\ \text{Or} & 226.981+74.42-201.3-90=10.101\end{array}\right\}$$

$$\left.\begin{array}{ll}\text{Also} & (6.2)^3+2(6.2)^2-33(6.2)-90=\text{2d error}\\ \text{Or} & 238.328+76.88-204.6-90=20.608\end{array}\right\}$$

$$\text{Difference of errors}=20.608-10.101=10.507.$$

Therefore, $10.507 : 0.1 :: 10.101 : 0.096$, the correction.

$6.1-0.096=6.004$. Let $x=6.004$ or $6.005$.

$$\left.\begin{array}{ll}\text{Then} & (6.004)^3+2(6.004)^2-33(6.004)-90=\text{1st error}\\ \text{Or} & 216.432+72.096-198.132-90=0.396\end{array}\right\}$$

$$\left.\begin{array}{ll}\text{Also} & (6.005)^3+2(6.005)^2-33(6.005)-90=\text{2d error}\\ \text{Or} & 216.54+72.12-198.165-90=0.495\end{array}\right\}$$

$$\text{Difference of errors}=0.495-0.396=0.099.$$

Therefore, $0.099 : 0.001 :: 0.396 : 0.004$, the correction.

$6.004-0.004=6$, which satisfies the equation, and is a root.

$$(x^3+2x^2-33x-90)\div(x-6)=x^2+8x+15$$

This quotient equated with zero is the equation which contains the remaining roots. But from this equation, by the Rule given on page 57, we have,

$$x=-4\pm\sqrt{-15+16}=-4\pm1=-5 \text{ or } -3.$$

Therefore the roots of the given equation are $-3$, $-5$, 6.

Ex. 5. What is a near value of one of the roots of the equation

$$x^3+9x^2+4x=80\,?$$

Transpose 80, and suppose one of the roots to be 2.5 or 2.6.

$$\left.\begin{array}{ll}\text{Then} & (2.5)^3+9(2.5)^2+4(2.5)-80=\text{1st error}\\ \text{Or} & 15.625+56.25+10-80=1.875\end{array}\right\}$$

$$\left.\begin{array}{ll}\text{Also} & (2.6)^3+9(2.6)^2+4(2.6)-80=\text{2d error}\\ \text{Or} & 17.576+60.84+10.4-80=8.816\end{array}\right\}$$

Difference of errors $=8.816-1.875=6.941$.

Therefore, $6.941 : 0.1 :: 1.875 : 0.027$, the correction.

And $2.5-0.027=2.473$. Let $x=2.473$ or $2.474$.

$$\left.\begin{array}{ll}\text{Then} & (2.473)^3+9(2.473)^2+4(2.473)-80=\text{1st error}\\ \text{Or} & 15.124198+55.041561+9.892-80=0.057759\end{array}\right\}$$

$$\left.\begin{array}{ll}\text{Also} & (2.474)^3+9(2.474)^2+4(2.474)-80=\text{2d error}\\ \text{Or} & 15.142552+55.086084+9.896-80=0.124636\end{array}\right\}$$

Difference of errors $=0.124636-0.057759=0.066877$

Therefore,

$0.066877 : 0.001 :: 0.057759 : 0.00086$, the correction.

And $2.473-0.00086=2.47214$.

Substituting this number for $x$ in the given equation, we have

$$(2.47214)^3+9(2.47214)^2+4(2.47214)-80=\text{the error.}$$

Or $15.108424+55.003285+9.88856-80=0.000269$.

The error is very small, and hence $2.47214$ is a near value of one of the roots.

Ex. 6. What is a near value of one of the roots of the equation

$$x^3+x^2+x=100\,?$$

Transpose 100, and suppose one of the roots to be 4.3 or 4.4.

$$\left.\begin{array}{ll}\text{Then} & (4.3)^3+(4.3)^2+(4.3)-100=\text{1st error}\\ \text{Or} & 79.507+18.49+4.3-100=2.297\end{array}\right\}$$

$$\left.\begin{array}{ll}\text{Also} & (4.4)^3+(4.4)^2+(4.4)-100=\text{2d error}\\ \text{Or} & 85.184+19.36+4.4-100=8.944\end{array}\right\}$$

Difference of errors $=8.944-2.297=6.647$.

Therefore, $6.647 : 0.1 :: 2.297 : 0.035$ nearly.

And $4.3-0.035=4.265$. Let $x=4.265$ or $4.266$.

$$\left.\begin{array}{ll}\text{Then} & (4.265)^3+(4.265)^2+(4.265)-100=\text{1st error}\\ \text{Or} & 77.581309+18.190225+4.265-100=0.036534\end{array}\right\}$$

$$\left.\begin{array}{ll}\text{Also} & (4.266)^3+(4.266)^2+(4.266)-100=\text{2d error}\\ \text{Or} & 77.635893+18.198756+4.266-100=0.100649\end{array}\right\}$$

Difference of errors $=0.100649-0.036534=0.064115$.

Therefore,

$$0.064115 : 0.001 :: 0.036534 : 0.00057, \text{ the correction.}$$

$$\text{And } 4.265-0.00057=4.26443.$$

Substituting this number for $x$ in the given equation, we have,

$$(4.26443)^3+(4.26443)^2+(4.26443)-100=\text{the error.}$$

$$77.550208+18.185363+4.26443-100=0.000001 \text{ nearly.}$$

The error is very small, and hence 4.26443 is a near value of one of the roots of the equation.

## Newton's Method.

Ex. 2. What is a near value of one of the roots of the equation

$$x^3+10x^2+5x=2600.$$

Let $x=r-z$.

Then, $$(r-z)^3+10(r-z)^2+5(r-z)=2600$$

Expanding,

$$r^3-3r^2z+3rz^2-z^3+10r^2-20rz+10z^2+5r-5z=2600$$

Rejecting the terms which contain $z^2$ and $z^3$, we have

$$r^3-3r^2z+10r^2-20rz+5r-5z=2600$$

This reduced gives $$z=\frac{2600-5r-10r^2-r^3}{-5-20r-3r^2}.$$

Assuming $r=11$, then

$$z=\frac{2600-55-1210-1331}{-5-220-363}=\frac{4}{-588}=-0.0068.$$

And $x=r-z=11-(-0.0068)=11.0068$, which is nearly one of the roots of the equation.

Ex. 3. What are the roots of the equation

$$x^3+2x^2-11x=12\,?$$

Let $x=r-z$, then $(r-z)^3+2(r-z)^2-11(r-z)=12$.

Expand. $r^3-3r^2z+3rz^2-z^3+2r^2-4rz+2z^2-11r+11z=12$

Rejecting the terms which contain $z^2$ and $z^3$,

$$r^3-3r^2z+2r^2-4rz-11r+11z=12$$

This reduced gives $$z=\frac{12+11r-2r^2-r^3}{11-4r-3r^2}$$

Let $r=4$, then $$z=\frac{12+44-32-64}{11-16-48}=\frac{-40}{-53}=0.75+$$

And $$x=r-z=4-0.75+=3.25 \text{ nearly.}$$

Again, assume $r=3.25$, and substitute,

$$z=\frac{12+35.75-21.125-34.328125}{11-13-31.6875}=\frac{-7.703125}{-33.6875}=0.23 \text{ nearly}$$

And $$x=r-z=3.25-0.23=3.02$$

Now let $r$ be placed equal to 3.02, and substitute,

$$z=\frac{12+33.22-18.2408-27.543608}{11-12.08-27.3612}=\frac{-0.564408}{-28.4412}=0.02 \text{ nearly.}$$

And $$x=r-z=3.02-0.02=3$$

Assume $r=3$, then $$z=\frac{12+33-18-27}{11-12-27}=\frac{0}{-28}=0.$$

There is no correction, and hence 3 is a root.

To obtain the other roots, divide $(x^3+2x^2-11x-12)$ by $(x-3)$, and equate the quotient with zero.

$$(x^3+2x^2-11x-12)\div(x-3)=x^2+5x+4=0$$

Now, by the Rule given on page 57, the roots of this equation are

$$x=-\frac{5}{2}\pm\sqrt{-4+\frac{25}{4}}=-\frac{5}{2}\pm\sqrt{\frac{9}{4}}=\frac{-5\pm3}{2}=-4 \text{ or } -1.$$

And the roots of the given equation are $-1$, $3$, $-4$.

Ex. 4. What are the roots of the equation

$$x^4+4x^3-7x^2-34x=24\,?$$

Let $x=r-z$, then $(r-z)^4+4(r-z)^3-7(r-z)^2-34(r-z)=24$

Expanding, $$r^4-4r^3z+6r^2z^2-4rz^3+z^4+4r^3-12r^2z+12rz^2-$$
$$4z^3-7r^2+14rz-7z^2-34r+34z=24$$

Rejecting the terms which contain powers of $z$,

$$r^4-4r^3z+4r^3-12r^2z-7r^2+14rz-34r+34z=24$$

This reduced gives $z=\dfrac{24+34r+7r^2-4r^3-r^4}{34+14r-12r^2-4r^3}$

Assume $r=4$,

Then $z=\dfrac{24+136+112-256-256}{34+56-192-256}=\dfrac{-240}{-358}=0.67+$

And $x=r-z=4-0.67=3.33$. Again, let $r=3.33$, then

$$z=\frac{24+113.22+77.6223-147.704148-122.96370321}{34+46.62-133.0668-147.704148}$$

$$=\frac{-55.82555121}{-200.150948}=0.28 \text{ nearly.}$$

And $x=r-z=3.33-0.28=3.05$.

Now let $r$ equal this corrected value of $x$; then

$$z=\frac{24+103.7+65.1175-113.4905-86.53650625}{34+42.7-111.63-113.4905}$$

$$=\frac{-7.20950625}{-148.4205}=0.05 \text{ nearly.}$$

Hence $x=r-z=3.05-0.05=3$

Assume $r=3$, then $z=\dfrac{24+102+63-108-81}{34+42-108-108}=\dfrac{0}{-140}=0$

There is no correction, and hence 3 is a root. In the given equation transpose 24, and divide the first member by $(x-3)$

$$(x^4+4x^3-7x^2-34x-24)\div(x-3)=x^3+7x^2+14x+8=0$$

Substitute $r-z$ for $x$, and the equation becomes

$$(r-z)^3+7(r-z)^2+14(r-z)+8=0$$

Expanding,

$$r^3-3r^2z+3rz^2-z^3+7r^2-14rz+7z^2+14r-14z+8=0$$

Rejecting the terms which contain $z^2$ and $z^3$,

$$r^3-3r^2z+7r^2-14rz+14r-14z+8=0$$

This reduced gives $z=\dfrac{8+14r+7r^2+r^3}{14+14r+3r^2}$

Suppose $r=-5$, then $z=\dfrac{8-70+175-125}{14-70+75}=\dfrac{-12}{19}=-0.63$

And $x=r-z=-5-(-0.63)=-4.37$. Again, let $r=-4.37$,

$$z=\frac{8-61.18+133.6783-83.453453}{14-61.18+57.2907}=\frac{-2.955153}{10.1107}=-0.3 \text{ nearly.}$$

Hence $x=r-z=-4.37-(-0.3)=-4.07$.

Suppose $r=-4.07$,

$$z=\frac{8-56.98+115.9543-67.419143}{14-56.98+49.6947}=\frac{-0.444843}{6.7147}=-0.07 \text{ nearly.}$$

And $x=r-z=-4.07-(-0.07)=-4$. Let $r=-4$,

$$z=\frac{8-56+112-64}{14-56+48}=\frac{0}{6}=0$$

Therefore $-4$ is a root. To obtain the two remaining roots, divide $(x^3+7x^2+14x+8)$ by $(x+4)$. The quotient is $x^2+3x+2$, and if placed equal to zero will give the required values of $x$. By the Rule given on page 57,

$$x=-\frac{3}{2}\pm\sqrt{-2+\frac{9}{4}}=-\frac{3}{2}\pm\sqrt{\frac{1}{4}}=\frac{-3\pm1}{2}=-2 \text{ or } -1.$$

Therefore the required roots are $-1$, $-2$, 3, $-4$.

## Horner's Method.

Ex. 9. Find the roots of the equation $x^3+5x^2+7x-47=0$.

If we represent the first member by $X$, and its derived polynomial by $X'$, we have

$$X'=3x^2+10x+7$$

To find $X''$, multiply $X$ by 3, and divide by $X'$.

$$\begin{array}{r|l}
3x^3+15x^2+21x-141 & 3x^2+10x+7 \\
\underline{3x^3+10x^2+\ \ 7x\qquad} & x+5 \\
5x^2+14x-141 &
\end{array}$$

Multiply this remainder by 3, and divide again.

$$\begin{array}{r}
15x^2+42x-423 \\
\underline{15x^2+50x+\ \ 35} \\
-\ 8x-458
\end{array}$$

Divide this remainder by 2; then $X''=4x+229$

To find $X'''$, multiply $X'$ by 4, and divide by $X''$.

$$\begin{array}{r|l}
12x^2+\ \ 40x+28 & 4x+229 \\
\underline{12x^2+687x\qquad} & 3x-647 \\
-647x+28 &
\end{array}$$

Multiply this remainder by 4, and divide again.

$$\begin{array}{r} -2588x+\quad 112 \\ -2588x-148163 \\ \hline 148275 \end{array} \quad \text{And } X'''=-148275$$

Collecting the values of $X$, $X'$, &c.

$$X\ =x^3+5x^2+7x-47, \qquad X''\ =4x+229,$$
$$X'=3x^2+10x+7, \qquad X'''=-148275.$$

When $x=-\infty$ the series of signs is $-+--$ giving 2 variations.
" $x=\ \infty$ " " $+++-$ " 1 "
" $x=\ 1$ " " $-++-$ " 2 "
" $x=\ 2$ " " $-++-$ " 2 "
" $x=\ 3$ " " $+++-$ " 1 "

Therefore the equation has only one real root, and its value is between 2 and 3.

The process for obtaining this root is as follows. The work, after the fifth transformation, is given in an abridged form.

| | | | |
|---|---|---|---|
| 1 | 5 | 7 | −47 (2.12383140 |
| | 7 | 21 | ${}_{1}$−5 |
| | 9 | ${}_{1}$39 | ${}_{2}$−989 |
| | ${}_{1}$11 | 4011 | ${}_{3}$−159872 |
| | 111 | ${}_{2}$4123 | ${}_{4}$−34720133 |
| | 112 | 414564 | ${}_{5}$−1311746728 |
| | ${}_{2}$113 | ${}_{3}$416832 | ${}_{6}$−58649114113 |
| | 1132 | 41717289 | ${}_{7}$−16878841135 |
| | 1134 | ${}_{4}$41751387 | −170725576 |
| | ${}_{3}$1136 | 4176048284 | |
| | 11363 | ${}_{5}$4176957932 | |
| | 11366 | 417699204629 | |
| | ${}_{4}$11369 | ${}_{6}$41770261606 | |
| | 113698 | 41770272978 | |
| | 113706 | ${}_{7}$4177028435 | |
| | ${}_{5}$113714 | 4177028889 | $\frac{170725576}{417702934}=40872486+$ |
| | 1137143 | 417702934 | |
| | 113714 | | |
| | ${}_{6}$11371 | | |
| | 1137 | | |
| | 113 | | |
| | ${}_{7}$113 | | |

$x=2.12383140408724\,86.$ *Ans*

Ex. 10. Find the roots of the equation

$$x^3+3x^2+5x-1881718027578170.433=0.$$

If Sturm's Theorem is applied, we have

$$X=x^3+3x^2+5x-a, \qquad X''=-4x+b$$
$$X'=3x^2+6x+5, \qquad X'''=-c.$$

Here $a$, $b$, and $c$ represent known quantities.

When $x=-\infty$ the series of signs is $-++-$ giving 2 variations.
" $x=\infty$ " " $++--$ " 1 "

Therefore two of the roots are imaginary. The first figure of the real root might be found by Sturm's Theorem; but this would render necessary the long operation of finding the values of $a$, $b$, and $c$ in the expressions given above. To find the first figure by trial would be tedious also; but the latter process may be rendered simple by putting $x=100000y$.

$$\text{Then } y^2=\frac{x^2}{(100000)^2}, \text{ and } y^3=\frac{x^3}{(100000)^3}.$$

Divide the given equation by $(100000)^3$, and substitute.

$$y^3+0.00003y^2+0.0000000005y-1.881718027578170433=0$$

Now if $y$ be placed equal to 1 and 2 successively, the first member of the equation will give results with contrary signs. Therefore 1 is the first figure of the root.

To find the remaining figures, proceed as follows.

$$\begin{array}{llll}
1 & .00003 & .0000000005 & -1.881718027578170433\,(1.2345 \\
 & 100003 & 10000300005 & {}_1-881688027078170433 \\
 & 200003 & {}_1 30000600005 & {}_2-153674826978170433 \\
 & {}_1 300003 & 36400660005 & {}_3-20805639963170433 \\
 & 320003 & {}_2 43200720005 & {}_4-2591440281170433 \\
 & 340003 & 44289729005 & -306343628420433 \\
 & {}_2 360003 & {}_3 45387738005 & \\
 & 363003 & 45535499205 & \\
 & 366003 & {}_4 45683420405 & \\
 & {}_3 369003 & 45701933055 & \\
 & 369403 & 45720448205 & \\
 & 369803 & & \\
 & {}_4 370203 & & \\
 & 370253 & & \\
 & 370303 & &
\end{array}$$

$$\frac{306343628, \text{\&c.}}{45720448205}=67$$

$y=1.234567$, and $x=123456.7$. *Ans.*

Ex. 11. Find the roots of the equation

$$x^3-27x-36=0.$$

Upon trial (Art. 553), it is found that the equation has a negative root between $-1$ and $-2$. Putting $\pm 0$ for the absent term, and changing the signs of the alternate terms, we have a new equation,

$$x^3\pm 0-27x+36=0,$$

which has a positive root equal to the negative root of the original equation. This positive root is found as follows.

$$\begin{array}{llll}
1 & \pm 0 & -27 & 36\,(1.445 \\
 & 1 & -26 & {}_{1}10 \\
 & 2 & {}_{1}{-24} & \quad {}_{2}944 \\
 & {}_{1}3 & -2264 & \quad {}_{3}105984 \\
 & 34 & {}_{2}{-2112} & \qquad {}_{4}2196125 \\
 & 38 & -209504 & \\
 & {}_{2}42 & {}_{3}{-207792} & \\
 & 424 & -20757575 & \dfrac{2196125}{20735925}=106 \\
 & 428 & {}_{4}{-20735925} & \\
 & {}_{3}432 & & \\
 & 4325 & & \\
 & 4330 & & x=1.445106
\end{array}$$

Therefore $-1.445106$ is a root of the given equation. It is now required to reduce this equation to a quadratic equation which shall contain the remaining roots. By Art. 541 the coefficient of the second term of any equation is equal to the sum of all the roots with their signs changed, and the known term is the product of all the roots after their signs have been changed. Therefore if the coefficient of the second term of the given equation (the place of the absent term being supplied by $\pm 0$) be diminished by the root found, with its sign changed, and the known term be divided by the same, the results will be, respectively, the coefficient of the second term and the known term of the required quadratic equation.

$$\pm 0-1.445106=-1.445106$$

And $$-36\div 1.445106=-24.911667$$

$$x^2-1.445106x-24.911667=0$$

This contains the other roots. By the Rule given on page 57,

$$x=0.722553\pm\sqrt{24.911667+(0.722553)^2}=0.722553$$
$$\pm\sqrt{24.911667+0.522082837809}=0.722553\pm5.043188$$
$$=5.765741 \text{ or } -4.320635.$$

Ex. 12. Find the roots of the equation

$$x^3-8169.5x^2-8169.5x-8170.5=0.$$

Let $1000y=x$; then $(1000)^2y^2=x^2$, and $(1000)^3y^3=x^3$.

Divide the given equation by $(1000)^3$, and substitute.

$$y^3-8.1695y^2-0.0081695y-0.0000081705=0.$$

Now it is evident by inspection that one root of this equation is between 8 and 9. To find this root, proceed as follows.

| | | | | |
|---|---|---|---|---|
| 1 | −8.1695 | −0.0081695 | −0.0000081705 | (8.1705 |
| | −1695 | −13641695 | $_1$−109133641705 | |
| | 78305 | $_1$612798305 | $_2$−46260761205 | |
| | $_1$158305 | 628728805 | $_3$−333785355 | |
| | 159305 | $_2$644759305 | | |
| | 160305 | 656099655 | | |
| | $_2$161305 | 667489005 | | |
| | 162005 | | | |
| | 162705 | | | |

$y=8.1705$, and $x=8170.5$

Now to obtain the quadratic equation which contains the remaining roots, apply the method described in the solution of the last example. The result will be

$x^2+x+1=0$. But by the Rule given on page 57,

$$x=-\frac{1}{2}\pm\sqrt{-1+\frac{1}{4}}=-\frac{1}{2}\pm\sqrt{-\frac{3}{4}}=-\frac{1}{2}\pm\frac{1}{2}\sqrt{-3}$$

And the required roots are

$$8170.5,\ -\frac{1}{2}+\frac{1}{2}\sqrt{-3},\ -\frac{1}{2}-\frac{1}{2}\sqrt{-3}.$$

Ex. 13. Find the roots of the equation

$$x^3-48447647 1864=0.$$

The first figure of the root is readily found by the Rule for extracting the cube root of numbers. Then the operation is as follows.

| | | | |
|---|---|---|---|
| 1 | 0 | 0 | $-484476471864\,(7854$ |
| | 7 | 49 | $_{1}-141476471864$ |
| | 14 | $_{1}147$ | $_{2}-9924471864$ |
| | $_{1}21$ | 16444 | 739846864 |
| | 218 | $_{2}18252$ | |
| | 226 | 1836925 | |
| | $_{2}234$ | $_{3}1848675$ | |
| | 2345 | 184961716 | $x=7854$ |
| | 2350 | | |
| | $_{3}2355$ | | |
| | 23554 | The other roots are imaginary. | |

Ex. 14. Find the roots of the equation

$$x^3+x^2+x-13099751099=0.$$

If Sturm's Theorem is applied, we have

$$X=x^3+x^2+x-a,\qquad X''=-4x+b$$
$$X'=3x^2+2x+1,\qquad X'''=-c.$$

Here $a$, $b$, and $c$ represent known quantities.

When $x=-\infty$ the series of signs is $-++-$ giving 2 variations.
" $x=\infty$ " " $++--$ " 1 "

Therefore two of the roots are imaginary. To find the real root,

Let $1000y=x$; then $(1000)^2y^2=x^2$, and $(1000)^3y^3=x^3$.

Divide the given equation by $(1000)^3$, and substitute.

$$y^3+0.001y^2+0.000001y-13.099751099=0.$$

The first figure of the root is evidently 2. To find the remaining figures proceed as follows.

| | | | |
|---|---|---|---|
| 1 | 0.001 | 0.000001 | $-13.099751099\,(2.357$ |
| | 2001 | 4002001 | $_{1}-5095749099$ |
| | 4001 | $_{1}12004001$ | $_{2}-927458799$ |
| | $_{1}6001$ | 13894301 | $_{3}-116351249$ |
| | 6301 | $_{2}15874601$ | |
| | 6601 | 16222151 | |
| | $_{2}6901$ | $_{3}16572201$ | |
| | 6951 | 16621607 | |
| | 7001 | | |
| | $_{3}7051$ | | $y=2.357$, and $x=2357$. |
| | 7058 | | |

Ex. 15. Find the roots of the equation

$$x^4+5x^3+4x^2+3x-105=0.$$

Here, in order to apply Sturm's Theorem, we obtain

$$X=x^4+5x^3+4x^2+3x-105, \qquad X''=43x^2+4x+1695,$$
$$X'=4x^3+15x^2+8x+3, \qquad X'''=2909x+11048,$$
$$X''''=-1946318839.$$

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| When | $x=-\infty$ | the series of | signs is | $+-+--$ | giving | 2 | variat's. |
| " | $x=\infty$ | " | " | $++++-$ | " | 1 | " |
| " | $x=1$ | " | " | $-+++-$ | " | 2 | " |
| " | $x=2$ | " | " | $-+++-$ | " | 2 | " |
| " | $x=3$ | " | " | $++++-$ | " | 1 | " |
| " | $x=-1$ | " | " | $-+++-$ | " | 2 | " |
| " | $x=-3$ | " | " | $-+++-$ | " | 2 | " |
| " | $x=-5$ | " | " | $--+--$ | " | 2 | " |
| " | $x=-6$ | " | " | $+-+--$ | " | 3 | " |

Hence there are two real roots: a positive one between 2 and 3, and a negative one between $-6$ and $-5$.

To find the positive root, proceed as follows. The work is presented in an abridged form.

| | | | | |
|---|---|---|---|---|
| 1 | 5 | 4 | 3 | $-105$ ( 2.2173388 |
| | 7 | 18 | 39 | $-27$ |
| | 9 | 36 | 111 | $-23744$ |
| | 11 | 58 | 123128 | $-100986219$ |
| | 13 | 6064 | 135792 | $-46782557079$ |
| | 132 | 6332 | 136453781 | $-53617463308959$ |
| | 134 | 6604 | 137116944 | $-1219004388804$ |
| | 136 | 661781 | 137582804703 | $-11425290952$ |
| | 138 | 663163 | 138049344252 | $-377729160$ |
| | 1381 | 664546 | 138069369160347 | |
| | 1382 | 66551529 | 1380893958168 | |
| | 1383 | 66648507 | 1380913980697 | |
| | 1384 | 66745534 | 138093400835 | |
| | 13847 | 6674969449 | 13809393491 | |
| | 13854 | 66753855 | 13809446898 | |
| | 13861 | 66758015 | 1380945224 | |
| | 13868 | 6675843 | 1380945758 | |
| | 138683 | 667588 | | |
| | 13868 | 667598 | | |
| | 1386 | 66759 | | |
| | 138 | 667 | | |
| | 138 | 667 | | |
| | 13 | | | |

$$\frac{377729160}{1380945758}=273529$$

$$x=2.2173388273529$$

To find the negative root, change the signs of the alternate terms, and we shall have a new equation, which contains a positive root equal to the negative root of the original equation. This positive root is found as follows.

| | | | | |
|---|---|---|---|---|
| 1 | −5 | 4 | −3 | −105 (5.1166 |
| | 0 | 4 | 17 | −20 |
| | 5 | 29 | 162 | −29949 |
| | 10 | 79 | 170051 | −120398859 |
| | 15 | 8051 | 178254 | −121381496064 |
| | 151 | 8203 | 179091141 | −127872863082864 |
| | 152 | 8356 | 179929824 | |
| | 153 | 837141 | 180434515656 | |
| | 154 | 838683 | 180939763584 | |
| | 1541 | 840226 | 180990349592856 | |
| | 1542 | 84115276 | 181040941169184 | |
| | 1543 | 84207988 | | |
| | 1544 | 84300736 | | |
| | 15446 | 8431001476 | | |
| | 15452 | 8431929388 | | |
| | 15458 | | | |
| | 15464 | | | |
| | 154646 | | | |
| | 154652 | | | |

$$\frac{127872863082}{181040941169}=70632$$

$$x=5.116670632$$

Therefore −5.116670632 is a root of the given equation.

Ex. 16. Find the roots of the equation

$$x^4-80x^3+1998x^2-14937x+5000=0.$$

By trial, according to the property stated in Art. 553, one root is found to exist between 0 and 1. To find this root apply Horner's method. The work is given below in an abridged form.

| | | | | |
|---|---|---|---|---|
| 1 | −80 | 1998 | −14937 | 5000 (35098 |
| | −797 | 197409 | −14344733 | 6965681 |
| | −794 | 195027 | −13759692 | 1339000625 |
| | −791 | 192654 | −13663561875 | 118069147 |
| | −788 | 19226025 | −13567628500 | 9556915 |
| | −7875 | 19186675 | −135659053 | |
| | −7870 | 19147350 | −135641822 | |
| | −7865 | 1914654 | −13564029 | |
| | −7860 | 191458 | −13563876 | |
| | −786 | 19145 | | |
| | −786 | 19145 | | |
| | −786 | 1914 | | |
| | −78 | | | |
| | −78 | | | |

$$\frac{9556915}{13563876}=70458$$

$$x=0.3509870458$$

By trial another root is found to lie between 12 and 13. To find this root proceed as follows.

| | | | | |
|---|---|---|---|---|
| 1 | −80 | 1998 | −14937 | 5000 (12756 |
| | −70 | 1298 | −1957 | −14570 |
| | −60 | 698 | 5023 | −4036 |
| | −50 | 198 | 5267 | −2986559 |
| | −40 | 122 | 5367 | −3405859375 |
| | −38 | 50 | 5339063 | −2335593512704 |
| | −36 | −18 | 5296132 | |
| | −34 | −3991 | 5291946125 | |
| | −32 | −6133 | 5287687500 | |
| | −313 | −8226 | 5287166706216 | |
| | −306 | −837175 | 5286634868864 | |
| | −299 | −851725 | | |
| | −292 | −866250 | | |
| | −2915 | −86798964 | | |
| | −2910 | −86972892 | | |
| | −2905 | | | |
| | −2900 | | | |
| | −28994 | | | |
| | −28988 | | | |

$$\frac{23355935}{5286634868}=4418$$

$$x=12.7564418$$

Two values of $x$ have now been obtained, and the given equation may be reduced to a quadratic equation which contains the remaining roots. Thus, for the reason stated in the solution of Ex. 11, we have

$$(-12.7564418-0.3509870)=-13.1074288$$

And $-80-(-13.1074288)=-66.8925712.$

Also, $5000\div12.7564418\times0.3509870=5000\div4.4773452=$
$1116.7331929.$

Therefore $x^2-66.8925712x+1116.7331929=0$

is the equation which contains the remaining roots.

Now, by the Rule given on page 57, we have

$$x=33.4462856\pm\sqrt{-1116.7331929+(33.4462856)^2}$$
$$=33.4462856\pm\sqrt{-1116.7331929+1118.65402043676736}$$
$$=33.4462856\pm1.3859309=34.8322165 \text{ or } 32.0603547$$

And the roots of the equation are $\begin{cases}0.3509870\\12.7564418\\32.0603547\\34.8322165\end{cases}$

Ex. 17. Find the roots of the equation

$$x^4-19x^3+123x^2-302x+200=0.$$

Upon trial one of the roots is found to be 4. Then

$$(x^4-19x^3+123x^2-302x+200)\div(x-4)=x^3-15x^2+63x-50=0$$

is the equation which contains the other roots. By trial the integral figure of one of these roots is found to be 1; and by Horner's Method we have

$$\begin{array}{llll}
1 & -15 & 63 & -50\,(102804 \\
 & -14 & 49 & \;-1 \\
 & -13 & 36 & \quad -284792 \\
 & -12 & 357604 & \qquad -1386048 \\
 & -1198 & 355212 & \\
 & -1196 & 35425744 & \\
 & -1194 & 35330352 & \\
 & -11932 & & x=1.02804 \\
 & -11924 & &
\end{array}$$

To reduce the cubic equation to a quadratic, divide $-50$ by $-1.02804$, and subtract $-1.02804$ from $-15$; the quotient is the known term, and the remainder the coefficient of the second term of the new equation. Hence we have

$$x^2-13.97196x+48.63624=0.$$

And, by the Rule given on page 57,

$$\begin{aligned}
x&=6.98598\pm\sqrt{-48.63624+(6.98598)^2}\\
&=6.98598\pm\sqrt{-48.63624+48.80391656}\\
&=6.98598\pm0.40948=7.39546 \text{ or } 6.57650.
\end{aligned}$$

$$\text{And the required roots are }\left\{\begin{array}{l}1.02804\\4.00000\\6.57650\\7.39546\end{array}\right.$$

Ex. 18. Find the roots of the equation

$$x^5-7x^4+15x^3-58x^2+44x-300=0.$$

To find the number of real roots, apply Sturm's Theorem. Let $X$ represent the first member of the given equation, and $X'$ its derived polynomial; then

$$X'=5x^4-28x^3+45x^2-116x+44.$$

To find $X''$, multiply $X$ by 5, and divide by $X'$.

The first remainder is $-7x^4+30x^3-174x^2+176x-1500$

Multiply this by 5, and divide again.

The remainder is $-46x^3-555x^2+68x-7192$

Therefore $X''=46x^3+555x^2-68x+7192$

To find $X'''$, multiply $X'$ by 46, and divide by $X''$.

1st Rem.$=-4063x^3+2410x^2-41296+2024$

Multiply this by 46, and divide again.

2d Rem.$=2365825x^2-2175900x+29314200$

Now as the operation is continued, it is found that the coefficients are constantly becoming larger and the multiplications longer. To shorten the process, divide the last remainder by 10000, and neglect the decimal figures.

Then $X'''=-236x^2+217x-2931$

To find $X''''$, multiply $X''$ by 236, and divide by $X'''$.

1st Rem.$=140962x^2-150874x+1697312$

Or, neglecting the last three figures, $141x^2-151x+1697$.

Multiply this by 236, and divide again.

2d Rem.$=-5039x-12779$

Or, dividing by 100, $X''''=50x+128$

To find $X'''''$, multiply $X'''$ by 50, and divide by $X''''$.

1st Rem.$=41058x-146550$

Neglect the last three figures, multiply by 50, and divide again.

2d Rem.$=-12548$, and $X'''''=12548$

Collecting the values of $X$, $X'$, &c.

$$X\ \ =x^5-7x^4+15x^3-58x^2+44x-300,$$
$$X'\ =5x^4-28x^3+45x^2-116x+44,$$
$$X''=46x^3+555x^2-68x+7192,$$
$$X'''=-236x^2+217x-2931,$$
$$X''''=50x+128, \qquad X'''''=12548.$$

When $x=-\infty$ the series of signs is $-\ +\ -\ -\ -\ +$ giv'g 3 varia's.

" $x=\ \infty$ " " $+\ +\ +\ -\ +\ +$ " 2 "

Hence there is only one real root. By trial the first figure of this root is found to be 6. Find the remaining figures by Horner's method, as follows.

| | | | | | |
|---|---|---|---|---|---|
| 1 | −7 | 15 | −58 | 44 | −300 ) 6.119 |
| | −1 | 9 | −4 | 20 | −180 |
| | 5 | 39 | 230 | 1400 | −3119069 |
| | 11 | 105 | 860 | 14880931 | −153150994649 |
| | 17 | 207 | 880931 | 15783025 | −868075943280401 |
| | 23 | 20931 | 902094 | 158755905351 | |
| | 231 | 21163 | 923490 | 159683728405 | |
| | 232 | 21396 | 925655351 | 1605224835735511 | |
| | 233 | 21600 | 927823054 | 1613630018297605 | |
| | 234 | 2165351 | 929993110 | | |
| | 235 | 2167703 | 931950187279 | | |
| | 2351 | 2170056 | 933909173566 | | |
| | 2352 | 2172410 | | | |
| | 2353 | 217453031 | | | |
| | 2354 | 217665143 | | | |
| | 2355 | | | | |
| | 23559 | | | | |
| | 23568 | | | | |

$$\frac{86807594}{161363}=538$$

$$x=6.119538$$

NOTE.—The method of finding $X''''$ and $X'''''$, pursued in the solution of Ex. 18, is not strictly accurate. It may be used sometimes to abridge the labor of multiplication, though always with care that the sign of the first term of $X''$, $X'''$, &c., be not changed by neglecting part of the coefficients.

◆

## APPLICATION OF ALGEBRA TO GEOMETRY.

PROB. 10. Let $ABC$ be the proposed triangle, $CD$ a perpendicular drawn from one of its angles to the opposite side, and $AD$ and $BD$ the segments of that side. Let $s=$ the sum of the sides $AC$ and $BC$; $p=$ the perpendicular $CD$: let also $2d=$ the difference, and $2x=$ the sum of the segments $AD$ and $BD$.

By Art. 218, $AD=x+d$, and $BD=x-d$.

By Euc. 1, 47, $AC=\sqrt{(x+d)^2+p^2}$, and $BC=\sqrt{(x-d)^2+p^2}$

By the conditions, $s=\sqrt{(x+d)^2+p^2}+\sqrt{(x-d)^2+p^2}$

Squaring both sides, and transposing,

$$2\sqrt{(x+d)^2+p^2}\times\sqrt{(x-d)^2+p^2}=s^2-2x^2-2d^2-2p^2$$

Let $s^2-2d^2-2p^2=2m$; then, by substituting and dividing by 2, we have

$$\sqrt{(x+d)^2+p^2}\times\sqrt{(x-d)^2+p^2}=m-x^2$$

Squaring both sides, and performing the multiplication indicated, the result is

$$(x^2-d^2)^2+2p^2(x^2+d^2)+p^4=m^2-2mx^2+x^4$$

Expand. $x^4-2d^2x^2+d^4+2p^2x^2+2p^2d^2+p^4=m^2-2mx^2+x^4$

Trans. and unit. terms, $(2m+2p^2-2d^2)x^2=m^2-2p^2d^2-d^4-p^4$

Dividing by coefficients of $x^2$, $x^2=\dfrac{m^2-2p^2d^2-d^4-p^4}{2m+2p^2-2d^2}$

Extracting the square root, $x=\sqrt{\dfrac{m^2-2p^2d^2-d^4-p^4}{2m+2p^2-2d^2}}$

Substituting the values of $p$, $d$, and $m$, as given by the conditions, we have $x=472.5$.

Hence $x+d=720=AD$, $x-d=225=DB$.

Also $AC=\sqrt{AD^2+CD^2}=780$, $BC=\sqrt{BD^2+CD^2}=375$.

By the figure, $AB=AD+BD=945$.

Therefore the required sides are 780, 375, 945.

PROB. 11. Let $ABC$ be a triangle right angled at $B$; $BD$ the perpendicular from the right angle upon the hypotenuse. Let the base $AB$ be denoted by $x$, $BD$ by $a$, the hypotenuse $AC$ by $y$, and the perimeter by $m$; then $m-x-y=BC$.

By Euclid, 1, 47, $y^2=x^2+(m-x-y)^2$

Expanding, $y^2=x^2+m^2-2mx-2my+x^2+2xy+y^2$

Trans. and unit. terms, $2my-2xy=2x^2-2mx+m^2$ (1)

By Euclid, 6, 8, $y:x::m-x-y:a$

Mult. extremes and means, $ay=mx-x^2-xy$ (2)

Multiplying by 2, $2ay=2mx-2x^2-2xy$ (3)

Adding (1) and (3), $2ay+2my=m^2$

Dividing by $2a+2m$, $y=\dfrac{m^2}{2a+2m}$

Subst. for $a$ and $m$ their values, $y=\dfrac{(720)^2}{2\times720+2\times144}$

Reducing, $y=\dfrac{518400}{1440+288}=300=AC$

Subst. this value in (2), $144\times300=720x-x^2-300x$

Trans. and unit. terms, $x^2-420x=-43200$

By the Rule given on page 57, $x=210\pm\sqrt{-43200+(210)^2}$

$$=210\pm\sqrt{-43200+44100}=210\pm30=240=AB.$$

Therefore $BC=m-x-y=720-300-240=180.$

PROB. 12. Let $x=$ one side of the square, and $d=$ the difference between a diagonal and a side; then $x+d=$the diagonal.

By Euclid, 1, 47, $(x+d)^2=2x^2$

Expanding, $x^2+2dx+d^2=2x^2$

Transposing and uniting terms, $x^2-2dx=d^2$

By the Rule given on page 57, $x=d\pm\sqrt{d^2+d^2}=d+d\sqrt{2}.$

PROB. 13. Let $ABC$ be a triangle in which is inscribed the square $DEFG$; let also $CI$, the altitude of the triangle, cut $DG$ in $H$: then $HI=$ a side of the square. Make $x=DG$, $b=AB$, and $h=CI$: then $CH=h-x$.

By similar triangles, $AB:CI::DG:CH$

Or, $b:h::x:h-x$

Multiplying extremes and means, $bh-bx=hx$

Transposing, changing signs, and div. by $b+h$, $x=\dfrac{bh}{b+h}$

PROB. 14. Let $a=$ one of the given sides, $c=$ the other, and $b=$ the bisecting line. Represent the base by $x$, and one of its segments by $y$: then $x-y$ will denote the other segment.

By Euclid, 6, 3, $x-y:y::a:c$

Multiplying extremes and means, $cx-cy=ay$ (1)

By Euclid, 6, B, $ac=xy-y^2+b^2$ (2)

Transposing in (1), and div. by $(a+c)$, $y=\dfrac{cx}{a+c}$

Substituting this value of $y$ in (2), $ac=\dfrac{cx^2}{a+c}-\dfrac{c^2x^2}{(a+c)^2}+b^2$

Clearing of fractions,

$$a^3c+2a^2c^2+ac^3=acx^2+c^2x^2-c^2x^2+a^2b^2+2ab^2c+b^2c^2$$

Trans. and unit. terms, $acx^2=a^3c-a^2b^2+2a^2c^2-2ab^2c+ac^3-b^2c^2$

Or, $acx^2=(a^2+2ac+c^2)\times(ac-b^2)$

Dividing by $ac$, and extract. the root, $x=\sqrt{\dfrac{(a+c)^2\times(ac-b^2)}{ac}}$

Removing a factor from under the radical sign,

$$x=(a+c)\times\sqrt{\frac{ac-b^2}{ac}}$$

PROB. 15. Let $ABC$ be the triangle right angled at $B$, and $FBED$ the inscribed square. Let $AC$, the hypotenuse, $=h$; $DF$, the side of the inscribed square, $=a$: let also $BC=x$, and $AB=y$.

By similar triangles, $BC:AB::BC-BE:DE$

Or, $x:y::x-a:a$

Multiplying extremes and means, $xy-ay=ax$

Transposing $ay$, $xy=ax+ay=a(x+y)$ (1)

By Euclid, 1, 47, $x^2+y^2=h^2$ (2)

Multiplying (1) by 2, and adding (1) and (2),

$$x^2+2xy+y^2=h^2+2a(x+y)$$

Transposing $2a(x+y)$, $(x+y)^2-2a(x+y)=h^2$

Regarding $(x+y)$ as the unknown quantity, its value may be found by the Rule given on page 57, and we shall have

$x+y=a\pm\sqrt{h^2+a^2}$ (3), and $y=a-x\pm\sqrt{h^2+a^2}$

Substituting this value of $y$ in (1),

$$x(a-x\pm\sqrt{h^2+a^2})=a(a\pm\sqrt{h^2+a^2})$$

Or, $x^2-(a\pm\sqrt{h^2+a^2})x=-a(a\pm\sqrt{h^2+a^2})$

By resolving,

$$x=\frac{1}{2}(a\pm\sqrt{h^2+a^2})\pm\sqrt{a(a\pm\sqrt{h^2+a^2})+\frac{1}{4}(a\pm\sqrt{h^2+a^2})^2}$$

$$=\frac{1}{2}a(a\pm\sqrt{h^2+a^2})\pm\sqrt{\frac{1}{4}h^2-\frac{1}{2}a^2\pm\frac{a}{2}\sqrt{h^2+a^2}}.$$

Now it is evident from equations (1) and (3), that the values of $y$ must be the same as those of $x$. Therefore if one of the values of either $x$ or $y$ be taken as one of the perpendicular sides, the other value will be the other side. Substituting the members of the problem, the lengths of the sides are found to be 28 and 21.

PROB. 16. Let $4x=$ the base, and $3x=$ the perpendicular.

Then, $4x\times 3x\div 2=6x^2=$ the area of the triangle.

But by the conditions, $6x^2-7x=$ the hypotenuse.

And by Euclid, 1, 47, $(4x)^2+(3x)^2=(6x^2-7x)^2$

Expanding, $16x^2+9x^2=36x^4-84x^3+49x^2$

Extracting the square root, $5x=6x^2-7x$

Dividing by $x$, $5=6x-7$, and $x=2$.

Therefore $4x=8$, and $3x=6$, the required sides.

PROB. 17. Let $x=$ the breadth of the walk. The area of the walk is evidently equal to the two pairs of equal parallelograms which surround the grass plat. The sides of one of these parallelograms are $x$ and $(2x+18)$; and the parallelogram on the opposite side of the plat is equal to it. The sides of a third parallelogram are $x$ and 12; and the opposite parallelogram is equal to it. Now the sum of the areas of these parallelograms is

$$2\times(2x+18)x+2(12\times x)=4x^2+60x$$

But by the conditions, $4x^2+60x=12\times 18=216$

Dividing by 4, and applying the Rule given on page 57,

$$x=-\frac{15}{2}\pm\sqrt{54+\left(\frac{15}{2}\right)^2}=-\frac{15}{2}\pm\sqrt{\frac{216+225}{4}}=$$

$$-\frac{15}{2}\pm\sqrt{\frac{441}{4}}=-\frac{15}{2}\pm\frac{21}{2}=3, \text{ the breadth of the walk.}$$

PROB. 18. Let $6x$ and $5x$ denote the sides of the field.

Then $6x\times 5x=30x^2=$ the area.

But by the conditions, $30x^2\div 6=125$

Or, $5x^2=125$, and $x^2=25$

Therefore $x=5$, $6x=30$, and $5x=25$.

PROB. 19. Let $ABC$ be the triangle, and $BCDE$ the parallelogram, having the side $BC$ common. Let $x = AB = BD$, and $y = CD$. Now $BC = 60$, and therefore the area of the parallelogram $= BC \times CD = 60y$; and the area of the triangle

$$= \frac{1}{2} BC \times AB = \frac{60x}{2} = 30x.$$

By the conditions, $30x : 60y :: 5 : 8$

Multiplying extremes and means, $240x = 300y$

Dividing by 240, $x = \frac{5y}{4}$ (1)

By Euclid, 1, 47, $BC^2 + CD^2 = BD^2$, or $60^2 + y^2 = x^2$ (2)

Substituting the value of $x$ in this equation, $60^2 + y^2 = \frac{25y^2}{16}$

Transposing, $60^2 = \frac{25y^2}{16} - y^2 = \frac{25y^2 - 16y^2}{16} = \frac{9y^2}{16}$

Extracting the square root, $60 = \frac{3y}{4}$, and $y = 80$

Substituting this value of $y$ in (1), $x = \frac{400}{4} = 100$

Therefore $60y = 60 \times 80 = 4800$, the area of the parallelogram.

And $30x = 30 \times 100 = 3000$, the area of the triangle.

PROB. 20. Let $x =$ a side of the base of the smaller vat: it is also equal to the height of the larger vat. Let $y =$ a side of the base of the larger vat: it will also denote the altitude of the smaller vat. Now the solid contents of each vat is equal to the product of the base by the altitude: therefore

The solid contents of the larger vat is $xy^2$
" " " smaller " $x^2y$

By the conditions, $x^2y : xy^2 :: 4 : 5$ (1)

And $xy^2 - 20 = x^2y$ (2)

Dividing the first couplet of (1) by $xy$, $x : y :: 4 : 5$

Multiplying extremes and means, $5x = 4y$, and $x = \frac{4y}{5}$ (3)

Substituting this value of $x$ in (2), $y^2 \times \frac{4y}{5} - 20 = \frac{16y^3}{25}$

Clearing of fractions, $100y^3-2500=80y^3$
Transposing and uniting terms, $20y^3=2500$
Reducing, $y^3=125$, and $y=5$
Subst. the value of $y$ in (3), $5x=4\times 5$, and $x=4$.

Therefore $4=$ the depth of the larger vat, and $5=$ the depth of the smaller vat.

Prob. 21. Let $ABC$ be the triangle, $AD$ a perpendicular drawn from one of its angles to the opposite side: let also $P$ be a point within the triangle, and $PE$, $PF$, and $PG$ be perpendiculars from $P$ upon the three sides. Represent these perpendiculars respectively by $a$, $b$, and $c$; and let $x$ denote half of one of the sides of $ABC$.

Now $AD=\sqrt{AB^2-BD^2}=\sqrt{4x^2-x^2}=\sqrt{3x^2}=x\sqrt{3}$.
The area of the triangle $ABC$ is $AD\times BD=x\times x\sqrt{3}$
" " $PAB$ " $x\times PE=ax$
" " $PAC$ " $x\times PF=bx$
" " $PBC$ " $x\times PG=cx$
But $PAB+PAC+PBC=ABC$
Therefore, $x\times x\sqrt{3}=ax+bx+cx$
Dividing by $x\sqrt{3}$, $x=\dfrac{a+b+c}{\sqrt{3}}$.

Prob. 22. Let $x=$ the breadth of the street: then $9x-3$ $=$ a side of the square. Now, as in Prob. 17, the area of the street is equal to the sum of two pairs of equal parallelograms

$$=(9x-3+2x)\times 2x+(9x-3)\times 2x=40x^2-12x.$$

Also the perimeter of the square $=4\times(9x-3)$.

By the conditions, $40x^2-12x-228=36x-12$
Transposing and uniting terms, $40x^2-48x=216$
Dividing by 40, $x^2-\frac{6}{5}x=\frac{27}{5}$
By the Rule given on page 57,

$$x=\frac{3}{5}\pm\sqrt{\frac{27}{5}+\frac{9}{25}}=\frac{3}{5}\pm\sqrt{\frac{135+9}{25}}=\frac{3}{5}\pm\frac{12}{5}=3.$$

Therefore $9x-3=27-3=24$, a side of the square; and $(24)^2=576$, the area of the square.

PROB. 23. Let $ABC$ be the triangle right angled at $B$; let also $AD$ and $CE$ be the bisecting lines. Represent $AD$ by $b$, and $CE$ by $a$; and let $BE=x$, and $BD=y$: then $AB=2x$, and $BC=2y$.

By Euclid, 1, 47, $4x^2+y^2=b^2$ (1)

" " $4y^2+x^2=a^2$ (2)

Transposing $4y^2$ in equation (2), $x^2=a^2-4y^2$

Substituting in (1), $4(a^2-4y^2)+y^2=b^2$

Transposing and uniting terms, $-15y^2=-4a^2+b^2$

Changing signs, and dividing by 15, $y^2=\frac{4a^2-b^2}{15}$

Extracting the square root, $y=\sqrt{\frac{4a^2-b^2}{15}}$

Substituting this value in (1), $4x^2+\frac{4a^2-b^2}{15}=b^2$

Clearing of fractions, $60x^2+4a^2-b^2=15b^2$

Transposing and uniting terms, $60x^2=16b^2-4a^2$

Dividing by 60, and extracting the root, $x=\sqrt{\frac{4b^2-a^2}{15}}$.

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Since it has been publicly announced that Prof. Stanley was preparing a Key to Day's Algebra, it is due to that gentleman to state that this book is entirely the work of other hands.

THE END.